Operator delocalization in disordered spin chains via Pauli-basis expansion

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We investigate operator delocalization (scrambling) by introducing the operator length, defined from the expansion in the Pauli basis. This quantity can be efficiently evaluated in MPO simulations and increases logarithmically in time in many-body localized systems.

To quantify scrambling, the out-of-time-order correlator (OTOC) has been introduced [1, 2]. It is defined as $F(t) = \langle \hat{W}^\dagger(t)\, \hat{V}^\dagger\, \hat{W}(t)\, \hat{V}\rangle$, where $\hat{V}$ and $\hat{W}$ are local operators whose supports lie at a given distance, and $\hat{W}(t) = \mathrm{e}^{i\hat{H}t}\, \hat{W}\, \mathrm{e}^{-i\hat{H}t}$ is the time-evolved version with some local Hamiltonian $\hat{H}$. The OTOC displays a propagating wavefront: ahead of the front it grows rapidly, typically exponentially, whereas behind it saturates to a constant value [3, 4]. In clean interacting systems this propagation occurs as a linear or power-law light cone [5–9], while in disordered interacting systems the propagation is very slow and the light cone is logarithmic [10–13]. This reflects the fact that in the latter case thermalization is hindered due to a phenomenon called many-body localization (MBL) [14] and for that reason quantum information delocalizes very slowly, and the system becomes never locally thermal.

We provide a different perspective on this operator delocalization phenomenon by introducing a length associated with an operator, based on its expansion in the Pauli operators. Specifically, in addition to the mass defined in [16], each Pauli-string operator is assigned a length, given by the position of the rightmost site differing from the identity. The length of a time-evolving operator is then obtained by averaging this Pauli-string quantity over the distribution defined by the squared scalar products of the operator and the Pauli strings. The evolution of such quantities provides a complementary perspective on the scrambling: as the operator delocalizes, it increasingly overlaps with more nonlocal Pauli strings, causing its length and mass to increase over time. This leads to a crucial advantage: in fact, both the operator length and the operator mass can be computed efficiently and exactly within the matrix product operator (MPO) representation of the time-evolved operator.

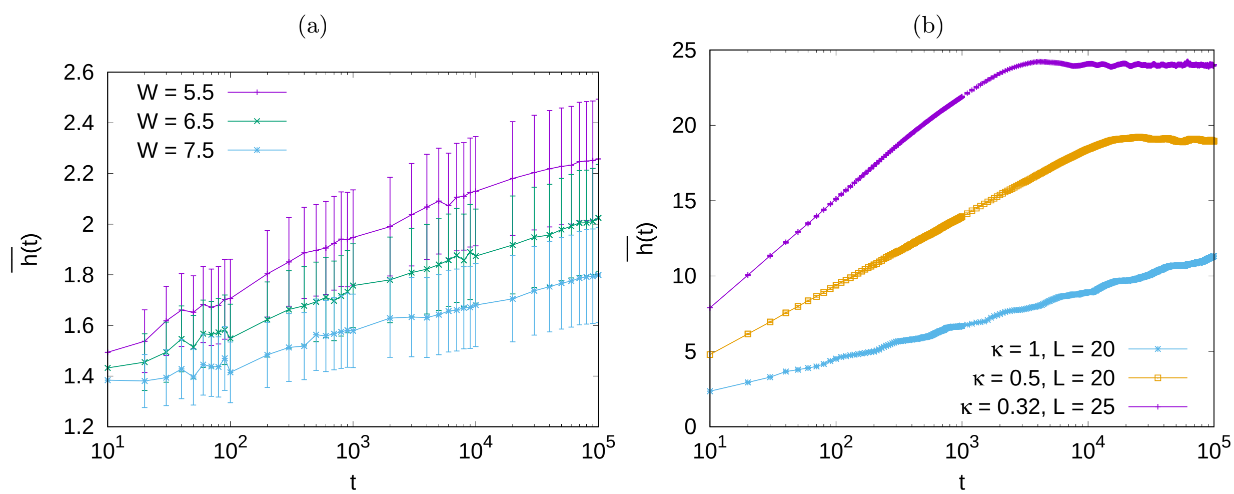

FIG. 1. (Panel a) The operator length $\overline{h}(t)$ averalged over disorder realizations as a function of the time t, for a disordered XXZ chain displaying many-body localization with $L = 12$ sites, as obtained by means of exact-diagonalization. Notice the logarithmic scale on the horizontal axis and the logarithmic growth in time over several decades. (Panel b) Same quantity for an ℓ-bit model for different values of the exponential decay rates κ of the couplings. Data are averaged over $N_\mathrm{r} = 192$ disorder realizations and we consider $W = 1$. Notice the logarithmic scale on the horizontal axis. The logarithmic increase in time is visible until saturation to the finite-size boundary value ($h \leq L$) occurs. For more details see [15].

We focus on the MBL case and find that the length of an initially localized operator grows logarithmically in time [Fig. 1(a)]. Looking at the scrambling from this perspective, the results are in full agreement with the OTOC wavefront propagating with a logarithmic light cone. Following this literature, we discuss a simple ℓ-bit model based on localized integrals of motion that provides the same logarithmic increase in time of the operator length [Fig. 1(b)]. We stress that, as occurs for the OTOC [17–24], our findings can be experimentally probed. In fact, our operator-marginal distributions can be accessed directly using standard tools available in current quantum-simulation platforms. The results we discuss in this contribution are available in the arXiv preprint [15].

[1] A. Larkin and Y. N. Ovchinnikov, Black holes and the butterfly effect, Sov. Phys. JETP **28**, 1200 (1969).
[2] S. H. Shenker and D. Stanford, Black holes and the butterfly effect, J. High Energ. Phys. **2014**, 67.
[3] S. Xu and B. Swingle, Accessing scrambling using matrix product operators, Nat. Phys. **16**, 199 (2019).
[4] B. Swingle, Unscrambling the physics of out-of-time-order correlators, Nat. Phys. **14**, 988 (2018).
[5] A. A. Patel, D. Chowdhury, S. Sachdev, and B. Swingle, Quantum butterfly effect in weakly interacting diffusive metals, Phys. Rev. X **7**, 031047 (2017).
[6] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Quantum entanglement growth under random unitary dynamics, Phys. Rev. X **7**, 031016 (2017).
[7] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Phys. Rev. X **8**, 021014 (2018).
[8] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws, Phys. Rev. X **8**, 031057 (2018).
[9] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation, Phys. Rev. X **8**, 031058 (2018).
[10] B. Swingle and D. Chowdhury, Slow scrambling in disordered quantum systems, Phys. Rev. B **95**, 060201 (2017).
[11] R. Fan, P. Zhang, H. Shen, and H. Zhai, Out-of-time-order correlation for many-body localization, Sci. Bull. **62**, 707 (2017).
[12] Y. Chen, Universal logarithmic scrambling in many body localization (2016), arXiv:1608.02765 [cond-mat.dis-nn].
[13] Y. Huang, Y. Zhang, and X. Chen, Out-of-time-ordered correlators in many-body localized systems, Ann. Phys. **529**, 10.1002/andp.201600318 (2016).
[14] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, Rev. Mod. Phys. **91**, 021001 (2019).
[15] J. Pineda, M. Collura, G. Passarelli, P. Lucignano, D. Rossini, and A. Russomanno, Operator delocalization in disordered spin chains via exact mpo marginals (2026), arXiv:2601.12446 [quant-ph].
[16] T. Schuster and N. Y. Yao, Operator growth in open quantum systems, Phys. Rev. Lett. **131**, 160402 (2023).
[17] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Measuring the scrambling of quantum information, Phys. Rev. A **94**, 040302 (2016).
[18] G. Zhu, M. Hafezi, and T. Grover, Measurement of many-body chaos using a quantum clock, Phys. Rev. A **94**, 062329 (2016).
[19] N. Y. Yao, F. Grusdt, B. Swingle, M. D. Lukin, D. M. Stamper-Kurn, J. E. Moore, and E. A. Demler, Interferometric approach to probing fast scrambling (2016), arXiv:1607.01801 [quant-ph].
[20] N. Yunger Halpern, Jarzynski-like equality for the out-of-time-ordered correlator, Phys. Rev. A **95**, 012120 (2017).
[21] N. Yunger Halpern, B. Swingle, and J. Dressel, Quasiprobability behind the out-of-time-ordered correlator, Phys. Rev. A **97**, 042105 (2018).
[22] M. Campisi and J. Goold, Thermodynamics of quantum information scrambling, Phys. Rev. E **95**, 062127 (2017).
[23] M. Gärttner, J. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Reay, Measuring out-of-time-order correlations and multiple quantum spectra in a trapped-ion quantum magnet, Nat. Phys. **13**, 781 (2017).
[24] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, Measuring out-of-time-order correlators on a nuclear magnetic resonance quantum simulator, Phys. Rev. X **7**, 031011 (2017).