# Topologies in Superconducting Josephson Devices

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# Outlook

***Role of topology in inhomogeneous networks***

1) **Free bosons**

2) **Copper pairs**

3) **Hard-core bosons**

4) **Limit of large n of O(n) models**

# Inhomogeneous network = non-translationally invariant network

Inhomogeneity due to topology (= how the lattice sites are connected) and/or to external fields

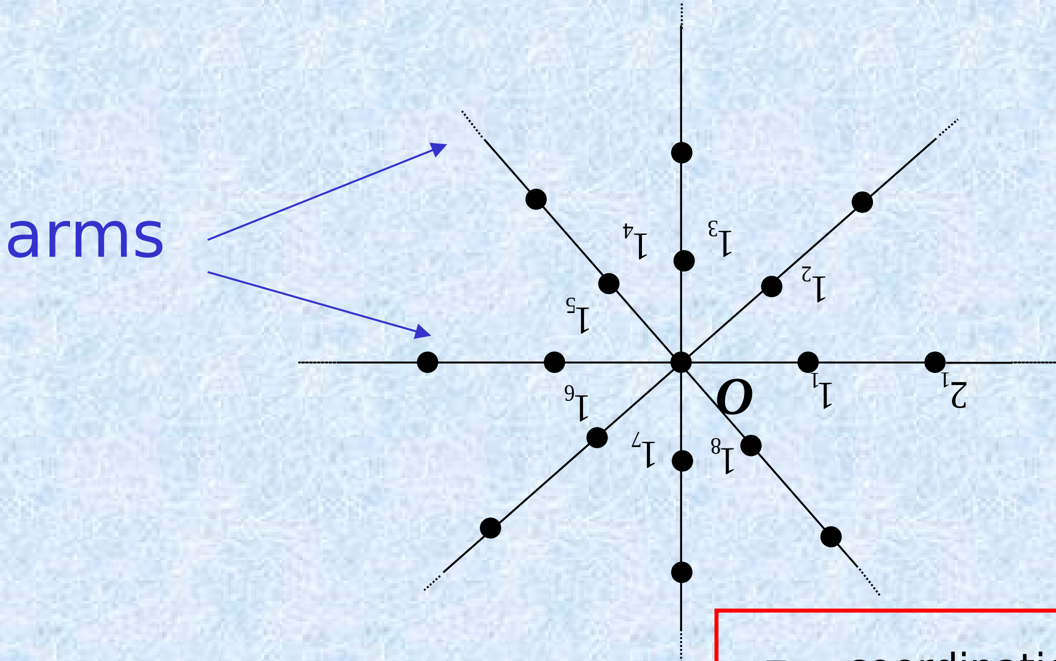
- **Long term goals**

To induce desired macroscopic coherent behaviors by acting on the topology of networks:

Enhance response of the system

Reduce effects of noise

# Free bosons on a star lattice




Total number of sites: $N_S = pL + 1$

| | |
|---|---|
| $z_i$ | coordination number of a given site: **2** |
| $z_O$ | coordination number of the center: ***p*** |

**Spatial Bose-Einstein condensation in the center at** $T < T_C$

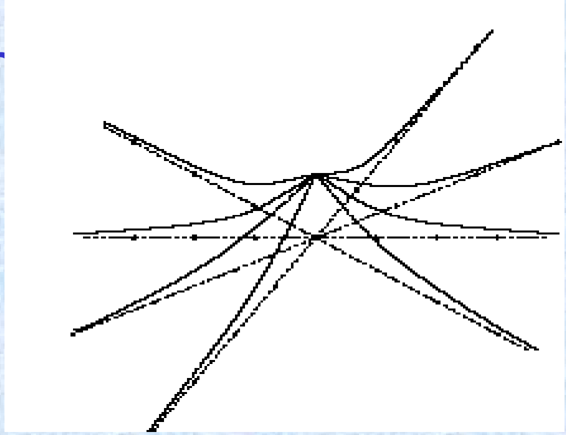
$$\hat{\mathbf{H}} = -t \sum_{i,j} A_{ij} \hat{\mathbf{a}}_i^{+} \hat{\mathbf{a}}_j$$

$$-t \sum_{j} A_{ij} \psi_\nu(j) = E_\nu \psi_\nu(i)$$

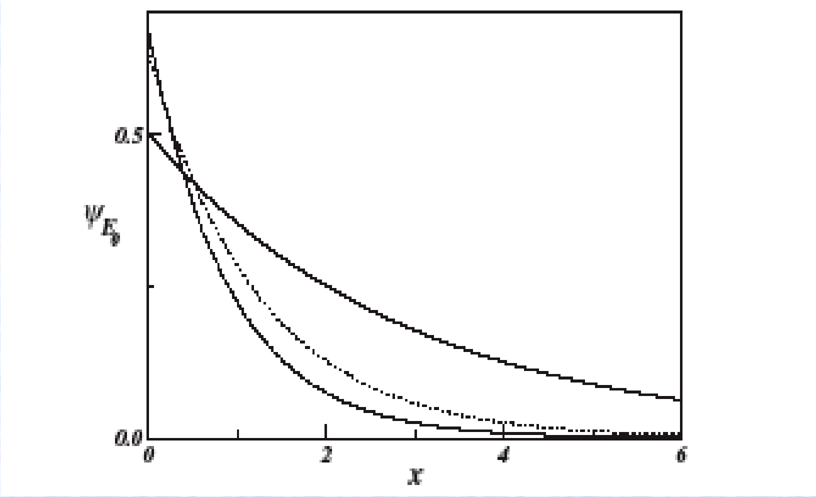
# Ground-state wavefunction

$$\psi_{E_0}(x) = \sqrt{\frac{p-2}{2p-2}}\, e^{-x/\xi_0} \qquad \xi_0 = \frac{2}{\log(p-1)}, \quad p > 2$$

Exponentially localized around the center, i.e. around the topological defect (~Anderson localization on inhomogeneous media)



Adding arms enhances localization



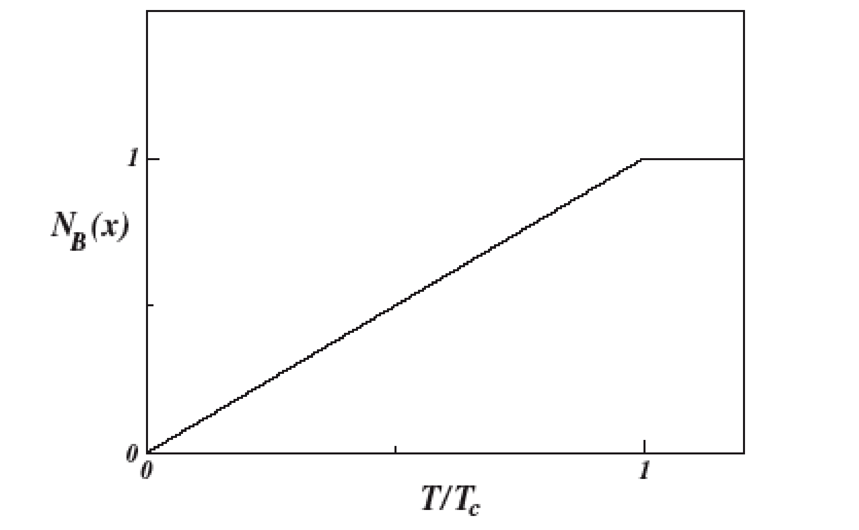
# Boson distribution

$$\frac{N_{E_0}\left( T/T_C \right)}{N_T} \approx 1 - \frac{T}{T_C} \qquad L \to \infty$$

$$T_C \approx \frac{p-2}{2\sqrt{p-1}} \frac{E_J}{k_B}$$

$$f = \frac{N_T}{N_S}$$

$$E_J = 2tf$$




Signature of the spatial Bose-Einstein condensation: decrease of the Josephson critical currents

$$\mathrm{N_B}\left( x >> 1; T/T_C \right) \approx f \frac{T}{T_C}$$

$x$ far away form the center

$$\frac{I_C^B\left(x >> 1; T/T_C\right)}{I_C^A} \approx \frac{T}{T_C}$$

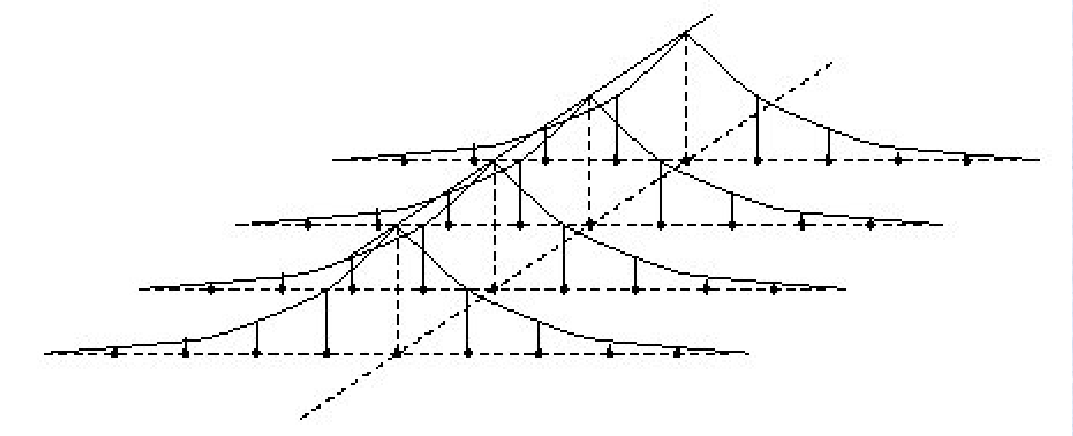
I. Brunelli et al., *J. Phys. B* **37**, S275 (2004)

# Free particles on a comb lattice

$$\hat{\mathrm{H}} = -t\sum_{\mathrm{i,j}} A_{\mathrm{ij}}\hat{\mathbf{a}}_{\mathrm{i}}^{+}\hat{\mathbf{a}}_{\mathrm{j}}$$

$$-t\sum_{\mathrm{j}} A_{\mathrm{ij}}\psi_{\nu}(j) = E_{\nu}\psi_{\nu}(i)$$

**Ground-state eigenfunction**



$$T_C \approx \frac{E_J}{k_B}$$

*Bose-Einstein critical temperature for free bosons*

$$\frac{I_c^B\left(x >> 1; T/T_C\right)}{I_c^A} \approx \frac{T}{T_C}$$

*far from (close to) the center: critical current decrease (enhancement)*

# Combs of Superconducting Josephson junctions

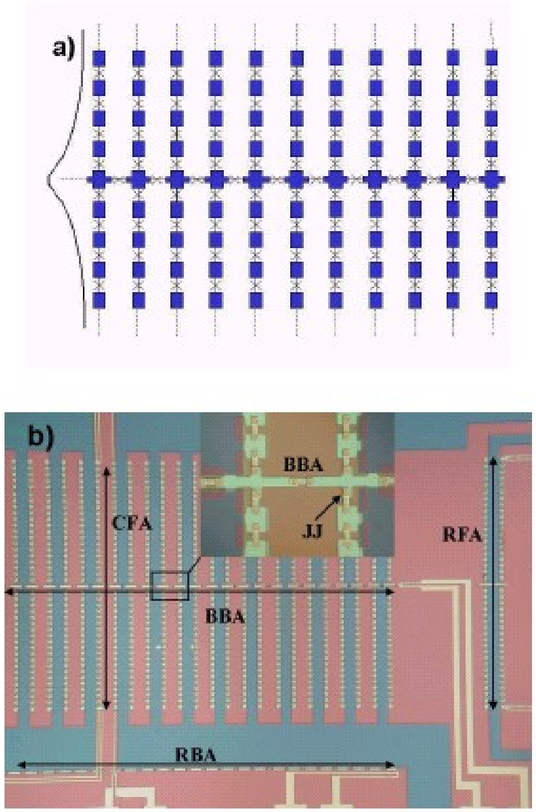


FIGURE 1

-Nb trilayer technology

-Josephson critical currents $I_C \sim 10\,\mu A$

- capacitance $C \sim 2\,pF$

- classical regime

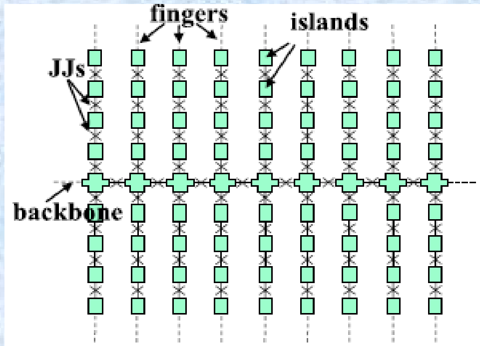
$$E_J = \frac{\hbar}{2e} I_C$$

$$E_c = \frac{e^2}{2C}$$

$$E_c / E_J < 0.001$$

P. Silvestrini *et al.*, Phys. Lett. A 370, 499 (2007)
P. Sodano *et al.*, New J. Phys. 8, 327 (2006)

# Bogoliubov-de Gennes theory for the critical current enhancement in comb shaped Josephson networks




## Inhomogeneous Comb

$i = (x, y)$

$x =$ *position on the backbone*

$y =$ *position on the finger*

$$A_{ij} = \delta_{xx'}(\delta_{y,y'+1} + \delta_{y,y'-1}) + \delta_{y0}\delta_{y'0}(\delta_{x,x'+1} + \delta_{x,x'-1})$$

## Homogeneous Chain

$i =$ *position on the chain*

$$A_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$$

## Spectrum

$$-t\sum_j A_{\mathrm{ij}}\psi_\alpha(j) = e_\alpha \psi_\alpha(i)$$

Ground state localized around the backbone –
“Hidden” spectrum of localized states

Planewave solutions

# Bogoliubov-de Gennes equations: continuous case

For an inhomogeneous fermionic systems with attractive interactions

$$H = H_0 + H_1$$

$$H_0 = \int dr \sum_{\sigma} \psi^{+}(r\sigma) h_0 \psi(r\sigma) \qquad H_1 = -\frac{V}{2} \int d\vec{r} \sum_{\sigma\sigma'} \psi^{+}(\vec{r}\sigma)\psi^{+}(\vec{r}\sigma')\psi(\vec{r}\sigma')\psi(\vec{r}\sigma)$$

$$h_0 = -\hbar^2 \nabla^2 / 2m + U_0(r) - \mu$$

$$\varepsilon_\alpha u_\alpha(\vec{r}) = \left[h_0 + U(\vec{r})\right] u_\alpha(\vec{r}) + \Delta(\vec{r}) v_\alpha(\vec{r})$$

$$\varepsilon_\alpha v_\alpha(\vec{r}) = -\left[h_0 + U(\vec{r})\right] v_\alpha(\vec{r}) + \Delta^*(\vec{r}) u_\alpha(\vec{r})$$

**Bogoliubov-de Gennes (BdG) Equations**

$$\Delta(\vec{r}) = V \sum_{\alpha} u_\alpha(\vec{r}) v_\alpha^*(\vec{r}) \tanh\left(\frac{\beta}{2}\varepsilon_\alpha\right)$$

$$U(\vec{r}) = -V \sum_{\alpha} \left[ |u_\alpha(\vec{r})|^2 f_\alpha + |v_\alpha(\vec{r})|^2 (1 - f_\alpha) \right]$$

**Self-consistency conditions**

$$f_\alpha = (e^{\beta\varepsilon_\alpha} + 1)^{-1}$$

# Bogoliubov-de Gennes Equations: lattice case

Discretization: $u_\alpha(r) = \sum_\alpha u_\alpha(i)\phi_i(r); \quad v_\alpha(r) = \sum_\alpha v_\alpha(i)\phi_i(r)$

$$\varepsilon_\alpha u_\alpha(i) = \sum_j \Gamma_{ij} u_\alpha(j) + \Delta(i) v_\alpha(i)$$

**Lattice BdG Equations**

$$\varepsilon_\alpha v_\alpha(i) = -\sum_j \Gamma_{ij} v_\alpha(j) + \Delta^*(i) u_\alpha(i)$$

$$\Gamma_{ij} = -t A_{ij} + U(i)\delta_{ij} - \tilde{\mu}\,\delta_{ij} \qquad \Delta(i) = \tilde{V}\sum_\alpha u_\alpha(i) v_\alpha^*(i)\tanh\left(\frac{\beta}{2}\varepsilon_\alpha\right)$$

**Encoding the network´s connectivity (=topology)**

**Self-consistency condition**

$t \approx -\int d\vec{r}\,\phi_i(\vec{r})\left(-\hbar^2\nabla^2/2m + U_0(\vec{r})\right)\phi_j(\vec{r})$ ← **Hopping parameter**

$\tilde{\mu} \approx \mu - \int d\vec{r}\,\phi_i(\vec{r})\left(-\hbar^2\nabla^2/2m\right)\phi_i(\vec{r})$ ← **Lattice chemical potential**

$\tilde{V} \approx V\phi_i^2(\vec{r} = \vec{r}_i)$

# Lattice Bogoliubov-de Gennes equations for the comb

Away from the backbone, the fingers may be regarded as a linear chain
($U(i)=U_c$ and $\Delta(i)=\Delta_c$). Setting on the backbone $U(i)=U_b$ and

$\Delta(i)=\Delta_b$, one gets with

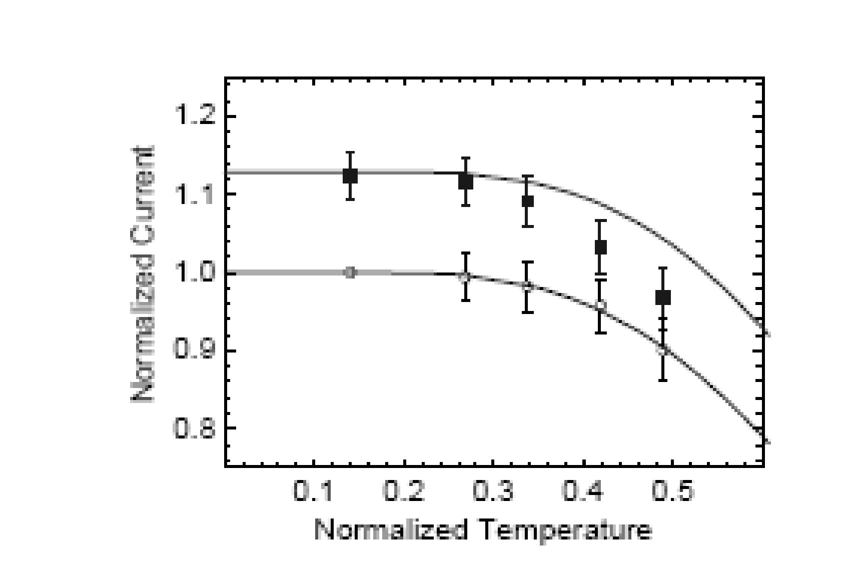
$$\Delta_b = \Delta_c + \frac{\Delta_b \tilde{V}}{\pi} \int_0^{\pi/2} dk \frac{\cos k}{\varepsilon_k \sqrt{1+\cos^2 k}} \tanh\left( \frac{\beta}{2} \varepsilon_k \right)$$

Contribution of the localized eigenstates of the adjacency matrix

At low temperature:

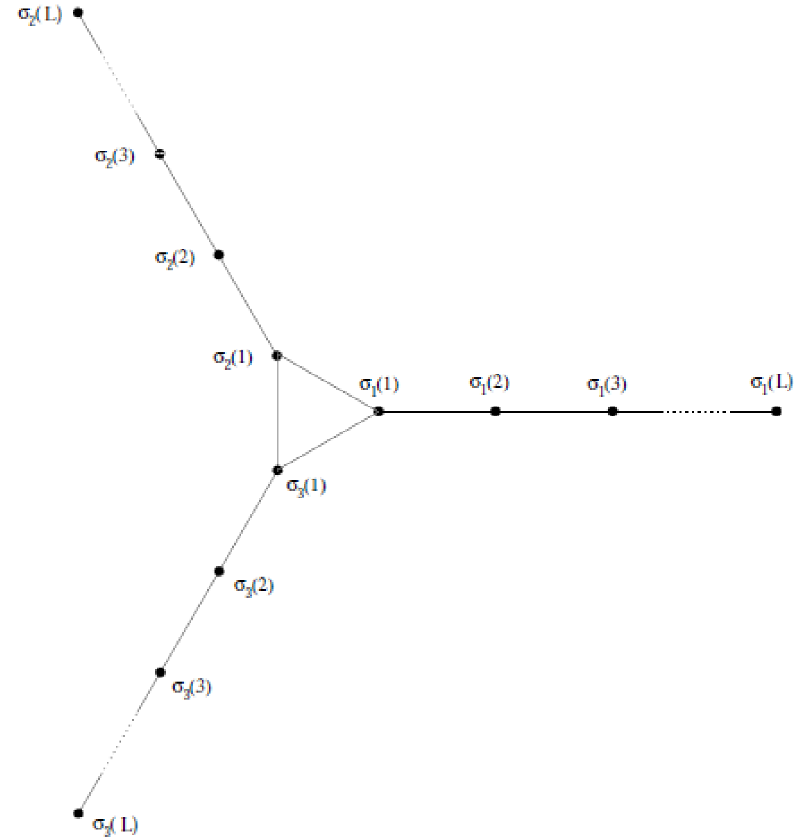
$$\frac{\Delta_b (T=0)}{\Delta_c (T=0)} = \frac{1}{1 - \frac{\eta \tilde{V}}{2\pi t}} \qquad \left( \eta = \frac{1}{\sqrt{2}} \log\left(1+\sqrt{2}\right) \right)$$

# Comparison for the critical currents with the experimental results



# Cooper pairs are hard-core bosons: XX model on the Y-junction (i.e., hard-core bosons on a Y-junction)

$$\widetilde{H}_3^{XX} = \sum_{j=1}^{L-1} \sum_{\alpha=1}^{3} \sigma_\alpha^+(j)\sigma_\alpha^-(j+1) + \rho \sum_{\alpha=1}^{3} \sigma_\alpha^+(1)\sigma_{\alpha+1}^-(1) + h.c.$$



**For a chain**

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**Jordan-Wigner transformations**

**But for a Y-junction?!**

# Proposed procedure to perform a Jordan-Wigner transformation giving raise to a local fermionic model (I)

*[N. Crampe' and A. Trombettoni, Nucl. Phys. B (2013)]*

Introduce an auxilary site "0"

$$H_3^{XX} = Id(0) \otimes \widetilde{H}_3^{XX}$$

$H_3^{XX}$ acting on the Hilbert space $\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes 3L}$

$H_3^{XX}$ acts as $\widetilde{H}_3^{XX}$ on the last $3L$ $\mathbb{C}^2$-spaces and trivially on the first $\mathbb{C}^2$-space

[see as well subsequent papers by A. Tsvelik in 2014-2015 for the Ising transverse field model at the critical point with the generalization to more than 3 legs]

# Procedure to perform a Jordan-Wigner transformation giving raise to a local fermionic model (II)

*[N. Crampe' and A. Trombettoni, Nucl. Phys. B (2013)]*

## Jordan-Wigner transformation

$$c_1(j) = \eta^x \left( \prod_{k=1}^{j-1} \sigma_1^z(k) \right) \sigma_1^-(j) \ , \quad c_2(j) = \eta^y \left( \prod_{k=1}^{j-1} \sigma_2^z(k) \right) \sigma_2^-(j) \ , \quad c_3(j) = \eta^z \left( \prod_{k=1}^{j-1} \sigma_3^z(k) \right) \sigma_3^-(j)$$

$$\eta^x = \sigma^x(0) \prod_{k=1}^{L} \sigma_2^z(k) \sigma_3^z(k) \ , \quad \eta^y = \sigma^y(0) \prod_{k=1}^{L} \sigma_1^z(k) \sigma_3^z(k) \ , \quad \eta^z = \sigma^z(0) \prod_{k=1}^{L} \sigma_1^z(k) \sigma_2^z(k)$$

*i)* the operators $c_\alpha(j)$ have to be fermionic

*ii)* the operator $\eta^a$ has to be $a$-th component of a spin operator

*iii)* the operators $c_\alpha(j)$ and the operators $\eta^a$ have to commute

Notice that for a spiral ordering the Hamiltonian would be not quadratic...

# Final result: a Kondo model

$$H_3^{XX} = -\sum_{j=1}^{L-1}\Big(\boldsymbol{c}(j)^\dagger\boldsymbol{c}(j+1)+\boldsymbol{c}(j+1)^\dagger\boldsymbol{c}(j)\Big) - \rho\,\boldsymbol{\eta}\,.\,\boldsymbol{c}(1)^\dagger\boldsymbol{S}\boldsymbol{c}(1)$$

$$\boldsymbol{c}(j)^\dagger = (c_1(j)^\dagger, c_2(j)^\dagger, c_3(j)^\dagger)\ ,\quad \boldsymbol{\eta} = (\eta^x,\eta^y,\eta^z)\ ,\quad \boldsymbol{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix}$$

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\ ,\qquad S^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\ ,\qquad S^z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4-channel Kondo model:

$$\chi_{imp} \propto T^{-1/3} \qquad \text{and} \qquad C_{imp} \propto T^{2/3}$$

**In the continuous limit the obtained Hamiltonian is integrable!**

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**"Topological" Kondo model**

[B. Beri and N. R. Cooper, PRL (2012)]

**Exact results**

At T=0 → A. Altland, B. Béri, R. Egger, and A. M. Tsvelik, J. Phys. A (2014)

At finite T → F. Buccheri, H. Babujian, V. E. Korepin, P. Sodano, and A. Trombettoni, Nucl. Phys. B (2015)

Exact results include:

junction ground-state energy

$$E_J^{(0)} = E_J^{(0)}(\lambda, M) = i \log \frac{i\Gamma\left(\frac{M+2}{4(M-2)} + \frac{i}{(M-2)\lambda}\right)\Gamma\left(\frac{3M-2}{4(M-2)} - \frac{i}{(M-2)\lambda}\right)}{\Gamma\left(\frac{M+2}{4(M-2)} - \frac{i}{(M-2)\lambda}\right)\Gamma\left(\frac{3M-2}{4(M-2)} + \frac{i}{(M-2)\lambda}\right)}$$

junction entropy at T=0

$$S_J^{(0)} = \log\sqrt{\frac{M}{2}} \qquad (\text{even } M)\,, \qquad\qquad S_J^{(0)} = \log\sqrt{M}$$

junction specific heat:

$$C_J \sim \left(\frac{T}{T_K}\right)^{\frac{2(M-2)}{M}}$$

$$T_K \simeq e^{-\frac{\pi}{\lambda(M-2)}}$$

and the junction free energy [not reported here]

Results available also for the anisotropic XY model in a transverse fields on a Y-junction *[D. Giuliano, P. Sodano, A. Tagliacozzo, and A. Trombettoni, NPB (2016)]*

# How to do it with ultracold atoms?

We need:

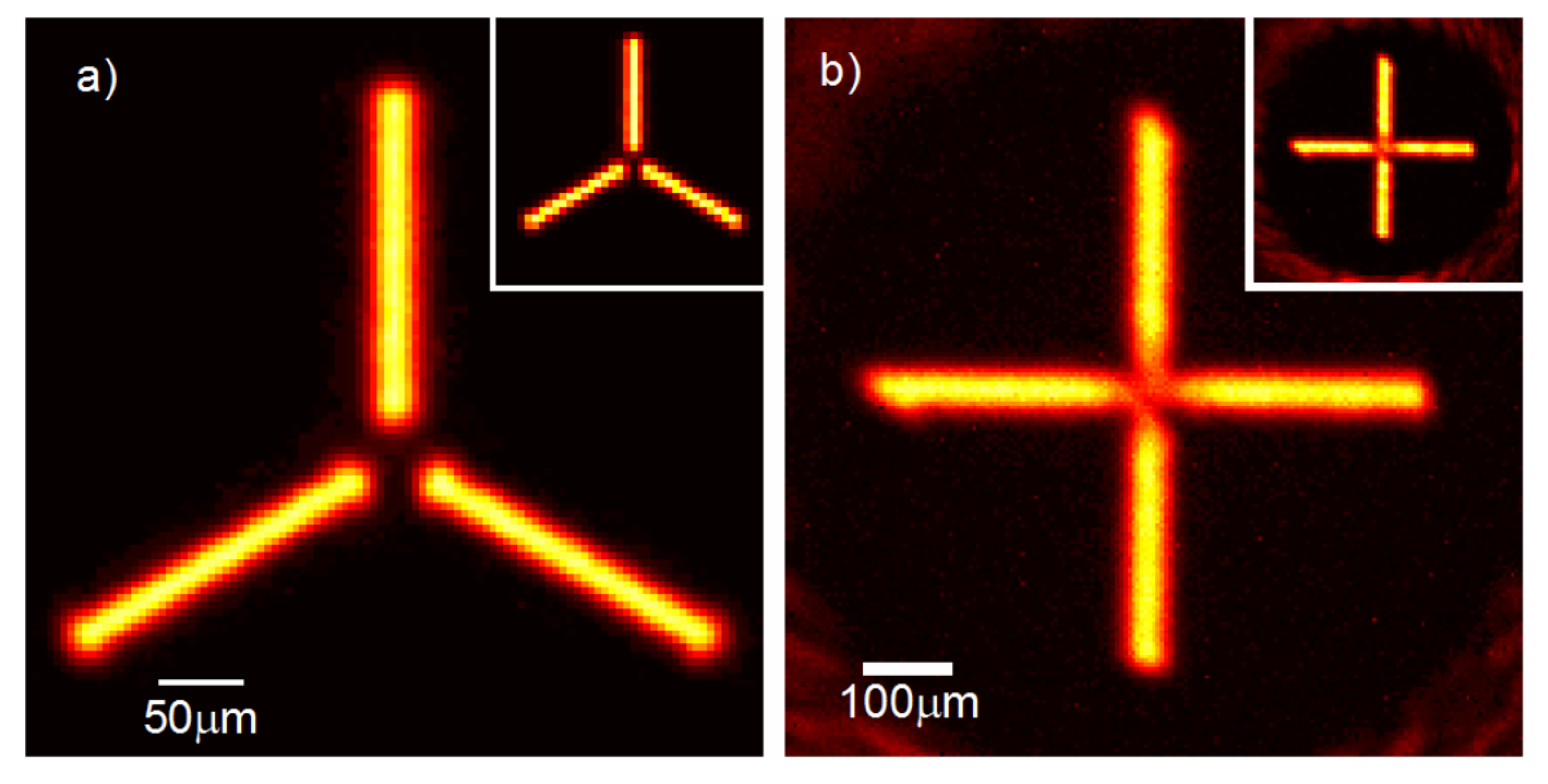
1) hard-core bosons (in the continuos limit) → Tonks-Girardeau gases

2)...we need as well a Y-junction!

Holographic traps provide an ideal tool to perform such geometries

# Holographic optical traps for atom-based topological Kondo devices

*[F. Buccheri, G. D. Bruce, A. Trombettoni, D. Cassettari, H. Babujian, V. E. Korepin, and P. Sodano, NJP (2016)]*




Kondo temperature with barriers of 2-3 microns is ~5-10 nK

# Outlook

***Role of topology in inhomogeneous networks***

1) **Free bosons**

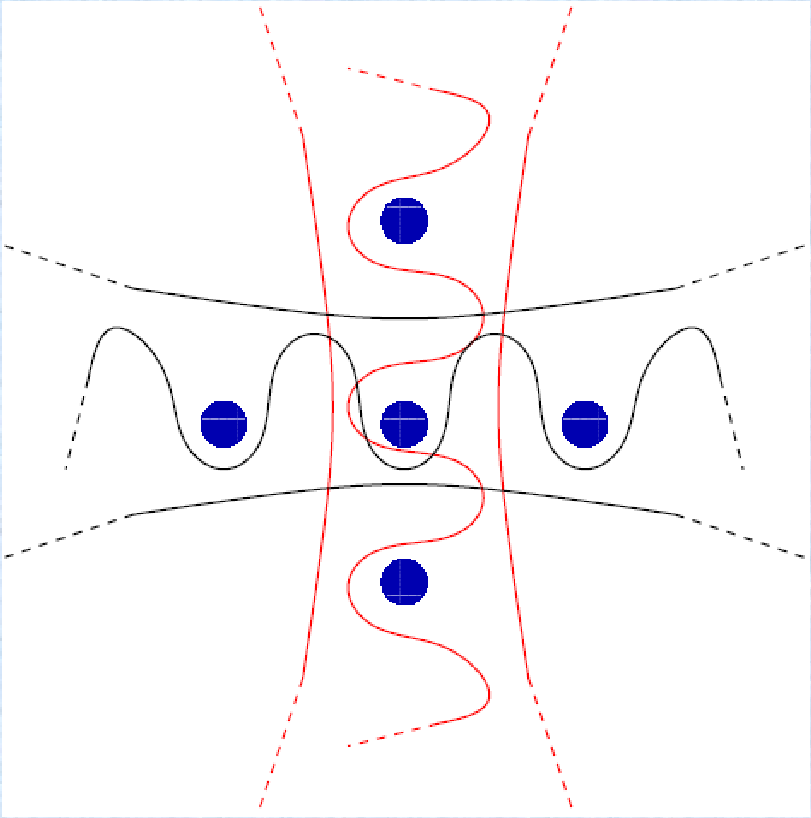
2) **Copper pairs**

3) **Hard-core bosons**

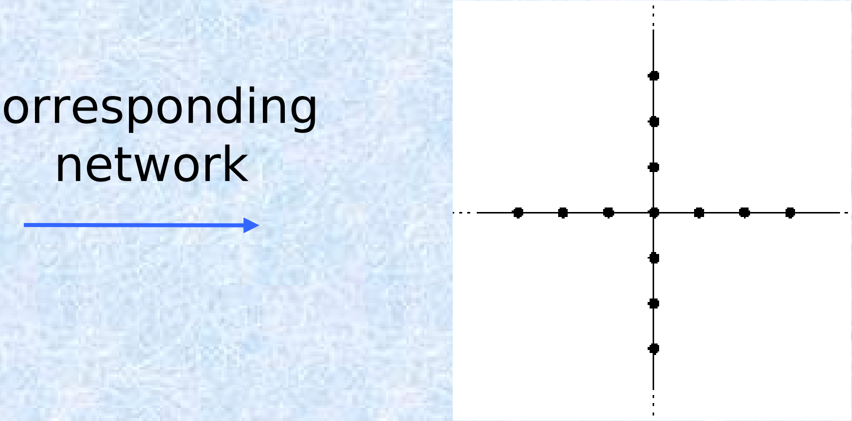
4) **Limit of large n of O(n) models → work in progress with Nikita Titov. Main result: the limit of large n, that for translationally invariant systems gives the free bosons, is not giving free bosons for inhomogeneous networks → the localized state contribution disappears**

**Thanks!**

# Creating a star-shaped network with ultracold bosons



Corresponding network



***Temperatures ~ 0-500 nK***

***Number of particles ~ 1000-10000***

***Number of wells ~ 100***

$$V(x) \approx V_0 \cos^2(kx)$$

$$k = {}^{2\pi}\!/_{\lambda} \qquad \lambda \approx 800 \text{ nm}$$

$$E_R = {}^{\hbar^2 k^2}\!/_{2m}$$

$$V_0 = s \cdot E_R \qquad s \approx 10 - 30$$

# Lattice Bogoliubov-de Gennes equations for the chain

$$i = 1, \ldots, N_S$$
$$\alpha \to k$$
$$E_k = -2t\cos k - \tilde{\mu} + U_c; \qquad \varepsilon_k = \sqrt{\Delta^2 + E_k^2}$$

$$1 = \frac{\tilde{V}}{4\pi t}\int_{-2t}^{2t} \frac{dE}{\sqrt{1 - \frac{E^2}{4t^2}}\sqrt{\Delta_c^2 + (E - \tilde{\mu} + U_c)^2}} \tanh\left(\frac{\beta}{2}\sqrt{\Delta_c^2 + (E - \tilde{\mu} + U_c)^2}\right)$$

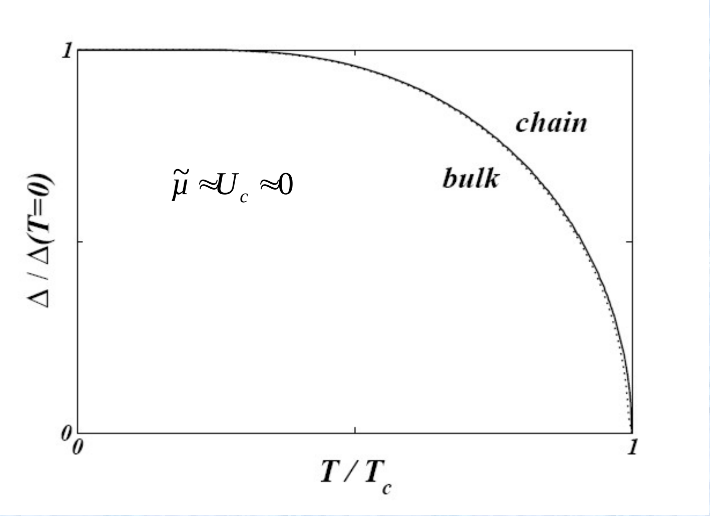
We have to set

$$\varepsilon_k \approx E_k$$
$$i.e., \; \tilde{\mu} \approx U_c \approx 0$$

One gets the „bulk" BCS results with

*Δ<<t*

$$\Delta_c(T=0) = 8t\, e^{-2\pi t/\tilde{V}}$$
$$k_B T_c = Ct\, e^{-2\pi t/\tilde{V}} \qquad (C \approx 4.54)$$
$$\frac{\Delta_c(T=0)}{k_B T_c} \approx 1.76$$



# Retrieving the standard BCS theory

In the homogeneous limit, the quantum number $\alpha$ is the momentum k:

$$\varepsilon_k = \sqrt{\Delta^2 + E_k^2} \qquad E_k = \hbar^2 k^2 / 2m - \mu + U$$

$$u_{\vec{k}}(\vec{r}) = L^{-3/2} U_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \qquad v_{\vec{k}}(\vec{r}) = L^{-3/2} V_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$$

$$U_{\vec{k}}^2 = \frac{1}{2}\left(1 + \frac{E_{\vec{k}}}{\varepsilon_{\vec{k}}}\right) \qquad V_{\vec{k}}^2 = \frac{1}{2}\left(1 - \frac{E_{\vec{k}}}{\varepsilon_{\vec{k}}}\right)$$

Putting $U=0$ and $\mu=E_F$ and assuming a BCS point-like interaction, one gets the BCS equation for the gap:

$$1 = \frac{n(0)V_{BCS}}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{dE}{\sqrt{E^2 + \Delta^2}} \tanh\left(\frac{\beta}{2}\sqrt{E^2 + \Delta^2}\right)$$

# Relation between the chemical potential and the Fermi energy

Using the equation for the number of particles

$$N = 2\sum_{\alpha} \int d\vec{r}\, |v_{\alpha}(\vec{r})|^2 + 2\sum_{\alpha} \int d\vec{r}\; f_{\alpha} \left( |u_{\alpha}(\vec{r})|^2 - |v_{\alpha}(\vec{r})|^2 \right)$$

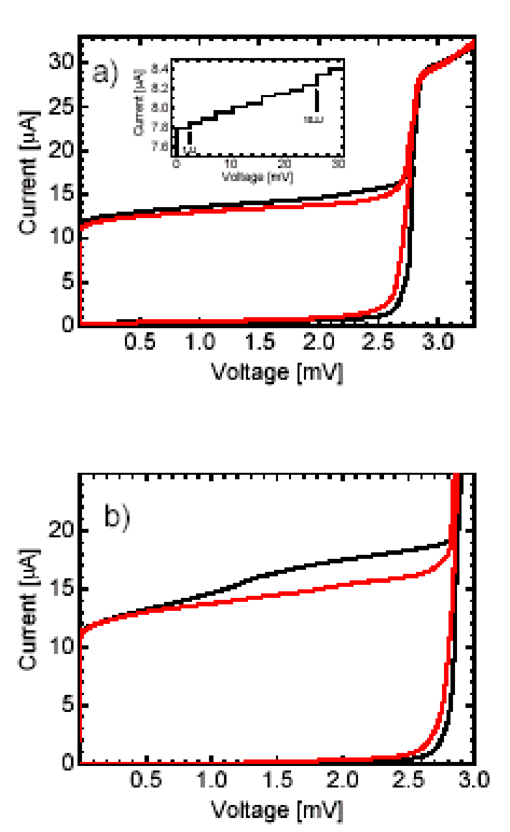
it follows at T=0 when $\Delta << E_F$

$$\mu - U \approx E_F$$

In general: since *U<0*, then $\mu < E_F$, increasing the attraction, the Hartree-Fock term *U* increases and the chemical potential μ decreases.

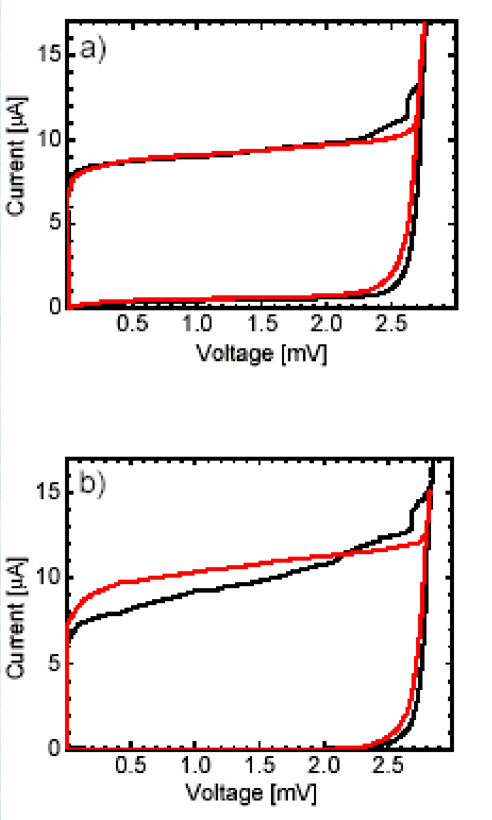
# Superconducting Josephson junctions on a comb lattice

On a comb of superconducting Josephson networks, one expects that critical currents along the backbone increase and along the fingers decrease:

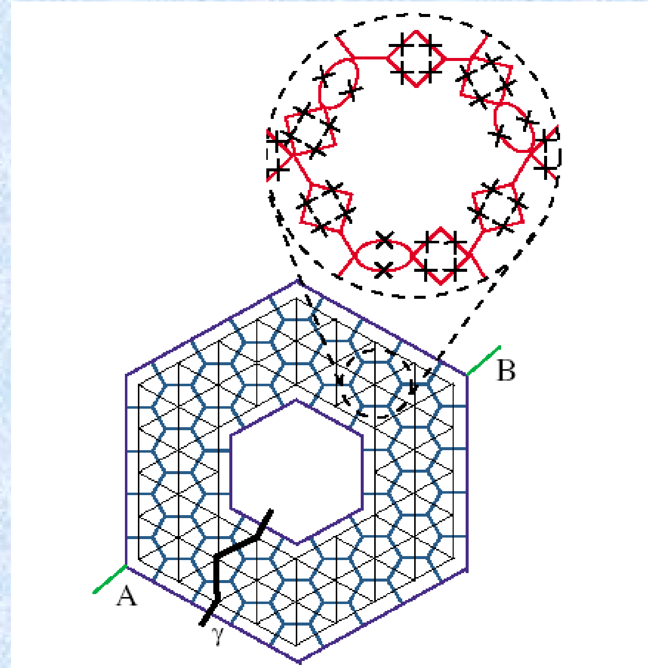



a) 4.2 K
b) 1.2 K

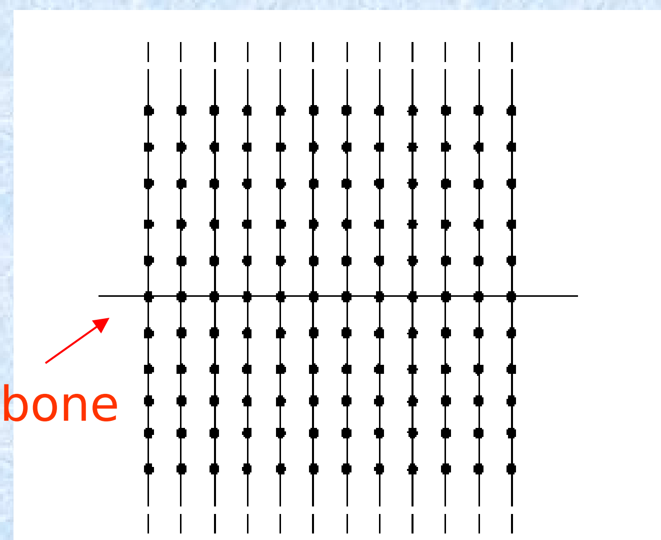
backbonefinger
vs. chain vs. chain



# Some examples of inhomogeneity effects (I




Ground states with high degeneracy
[B. Doucout et al., *PRL* **90**, 107003 (2003)]

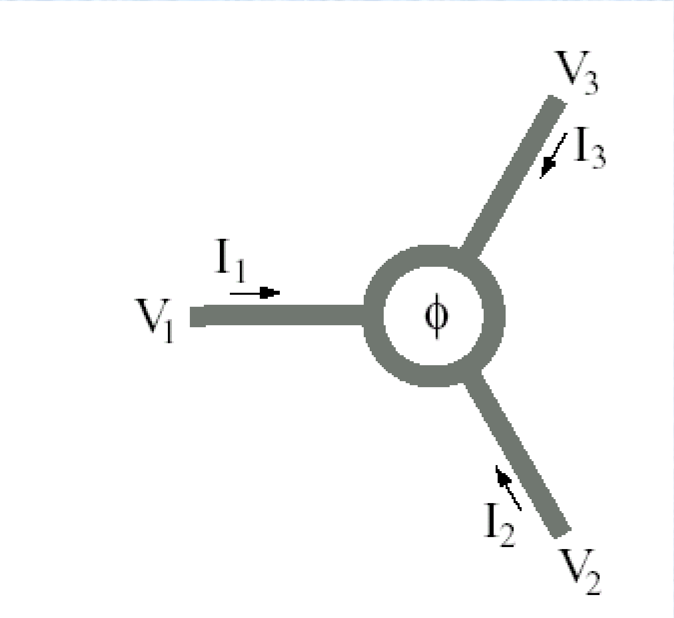


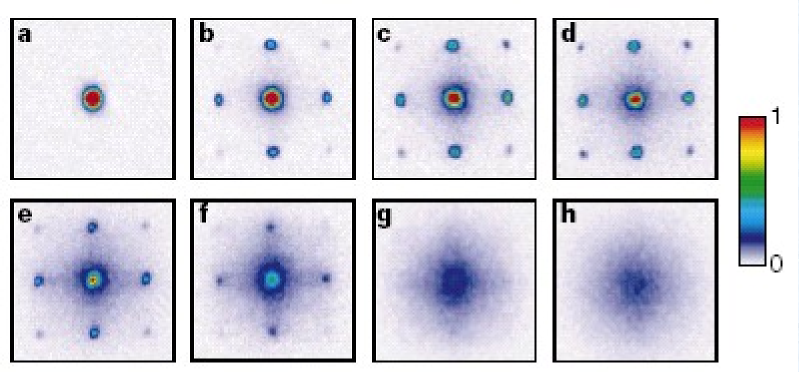

Free bosons undergo Bose-Einstein condensation: they localize on the comb's backbone
[R. Burioni et al., *EPL* **52**, 251 (2000)]

Superconducting grains: increasing the Josephson critical current along the backbone
[P. Sodano et., *New J. Phys.* **8**, 327 (2006)]

# Some examples of inhomogeneity effects (II)

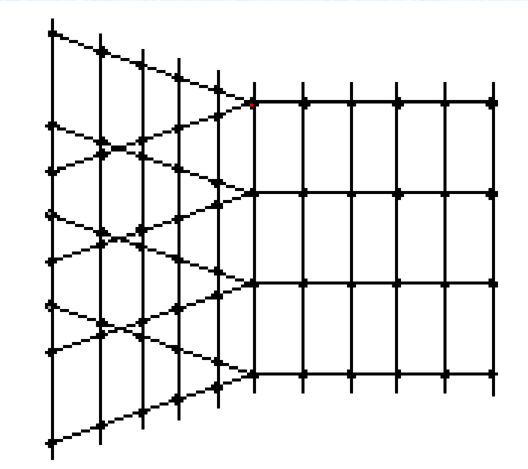
Junction of three quantum wires: new fixed points
[M. Oshikawa, C. Chamon, and I. Affleck, *PRL* **91**, 206403 (2003); *J. Stat. Mech.* **0602**, P008 (2006)]




“Wedding cake” of Mott domains surrounded by superfluid regions for bosons in a lattice + a magnetic trap
[M. Greiner et al., *Nature* **415**, 39 (2002) – Batrouni et al., *PRL* **89**, 117203 (2003)]

# ome examples of inhomogeneity effects (II



Critical behaviour at the junction of spin networks: local magnetization on a disordered bulk

[R. Marchetti, M. Rasetti, P. Sodano and A. Trombettoni, submitted]

# Superconducting weak links: a Josephson junction

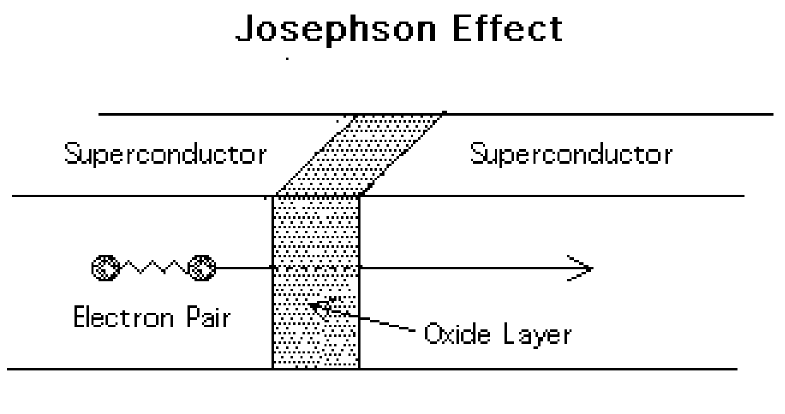



Fig. 13

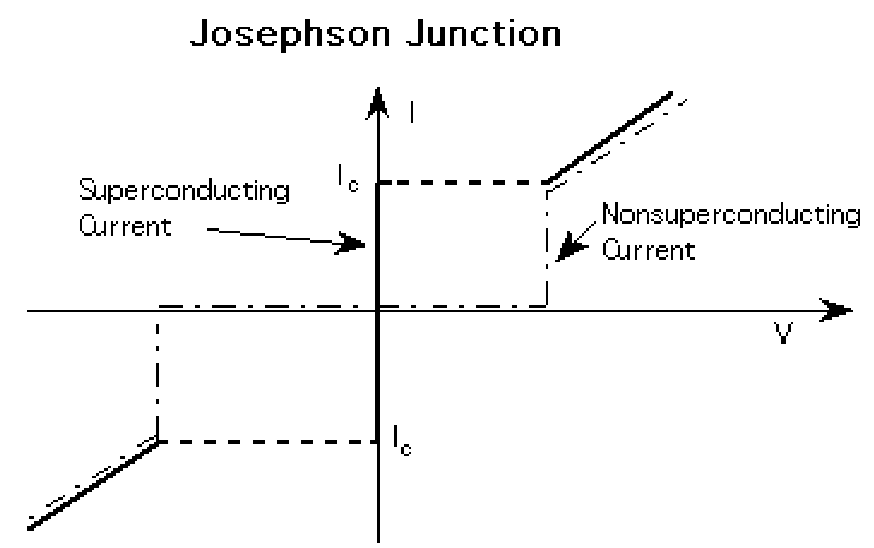



Fig. 14

**Josephson current at $T<T_{BCS}$**

# A superconducting Josephson junction

-) In absence of fields: $I = I_C \sin(\varphi_1 - \varphi_2)$

The critical current is proportional to the gap $\Delta$

In a SQUID the critical current can be tuned using a magnetic
eld:

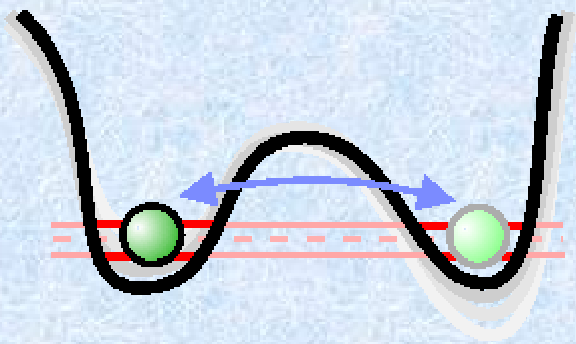
$$I_C = I_C(\Phi = 0) \left| \cos\left( \frac{\pi \Phi}{\Phi_0} \right) \right|$$

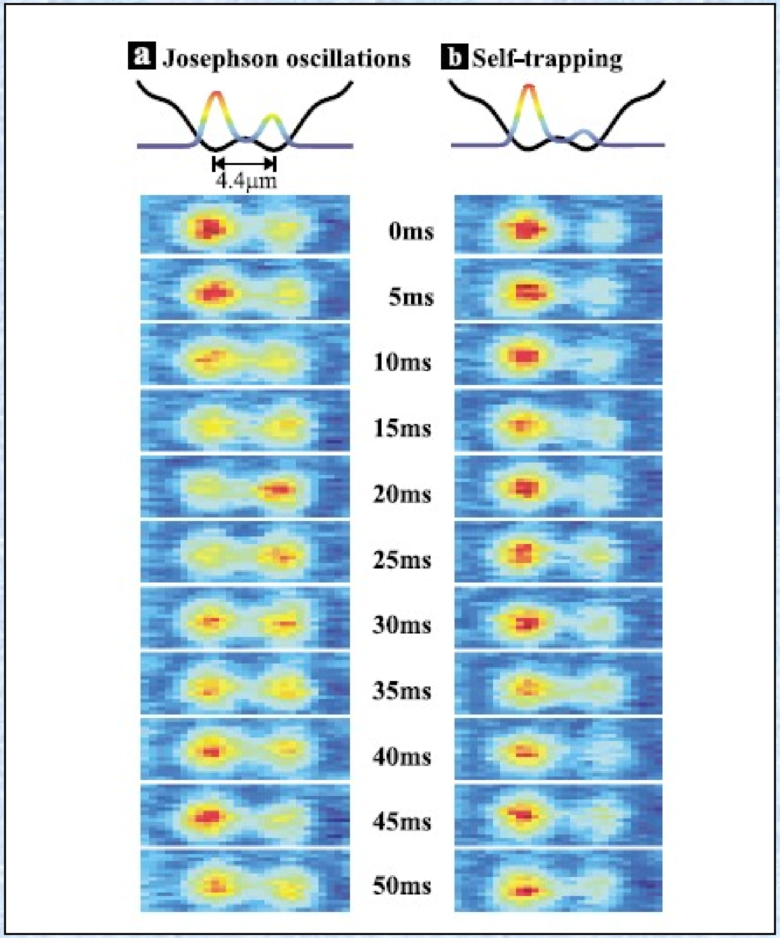
-) At finite temperature

$$\frac{I_C(T)}{I_C(0)} = \frac{\Delta(T)}{\Delta(0)} \tanh\left( \frac{\Delta(T)}{2 k_B T} \right)$$ *Ambegaokar-Baratoff*

# Analogies between bosonic and superconducting Josephson junctions

A Bose-Einstein condensate in a double well is a bosonic Josephson junction:






M. Albiez et al., PRL (2005)

# Theoretical models I: superconducting Josephson networks (Quantum Phase model)

$$\hat{\mathrm{H}} = \frac{1}{2}\sum_{\langle i,j\rangle} \hat{Q}_i \mathbf{C}^{-1}_{i,j} \hat{Q}_j - E_J \sum_{\langle i,j\rangle} \cos(\hat{\varphi}_i - \hat{\varphi}_j)$$

$$\left[\hat{\varphi}_i, \hat{n}_j\right] = i\delta_{i,j} \qquad \hat{Q}_j = 2e\,\hat{n}_j$$

$\mathbf{C}_{i,j}$ → **Capacitance matrix**

$E_c = e^2 \frac{C^{-1}_{i,i}}{2}$ → **Charging energy**

$E_J$ → **Josephson coupling**

$E_J >> E_c$ → **Classical XY regime**

$E_J << E_c$ → **Quantum XY regime**

# Theoretical models II: bosons in optical networks (Bose-Hubbard model)

$$\hat{\mathrm{H}} = \frac{1}{2}\sum_{<i,j>} \hat{n}_i U_{i,j} \hat{n}_j - (\mu - \varepsilon_i)\sum_i \hat{n}_i - \frac{t}{2}\sum_{<i,j>} \left(\hat{\mathbf{a}}_i^+ \hat{\mathbf{a}}_j + \hat{\mathbf{a}}_j^+ \hat{\mathbf{a}}_i\right)$$

$$\left[\hat{\mathbf{a}}_{\mathrm{i}}, \hat{\mathbf{a}}_{\mathrm{j}}^+\right] = \delta_{\mathrm{ij}} \qquad \hat{n}_j = \hat{\mathbf{a}}_j^+ \hat{\mathbf{a}}_j$$

$$\hat{\mathbf{a}}_i \approx \sqrt{n_i}\, e^{i\varphi_i} \quad \xrightarrow{\langle \hat{n}_j \rangle \text{ large}} \quad \text{Quantum Phase model for superconducting Josephson networks}$$

**In the following:** Josephson networks on discrete structures which are not necessarily regular lattices

# Mean-field analysis

Local order parameter:

$$\langle S_i \rangle = \tanh\left(\beta q J \langle S_i \rangle + \beta \Omega i^2\right)$$

$$\Omega i^2 << qJ \Rightarrow \langle S_i \rangle \approx \tanh\left(\beta q J \langle S_i \rangle\right)$$

usual mean-field equations, with magnetization at low temperatures

$$\Omega i^2 >> qJ \Rightarrow \langle S_i \rangle \approx \tanh\left(\beta \Omega i^2\right)$$

$$\Omega i_c^2 = qJ \Rightarrow \begin{cases} i < i_c & \langle S_i \rangle \sim (1 - T/T_C)^{1/2} \\ i >> i_c & \langle S_i \rangle \sim 1 \end{cases}$$

ood agreement with Monte Carlo results, even in d=2

**Is it possible to have a phase transition on a part of the network by controlling the inhomogeneity?** **Eventually when the remaining part is disordered, or when the dimension of this part is lower of the critical dimension…**

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# ISING MODEL WITH INTERACTION BETWEEN NONNEAREST NEIGHBORS

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A two-dimensional Ising lattice is considered in which, besides the usual interaction, there is an interaction along diagonals between nodes with equal row-plus-column parities. The free energy and the spontaneous magnetization are found as functions of the temperature. The form of the correlation function at large distances is derived at and close to the phase-transition point.

## 1. INTRODUCTION

THE Ising model consists of a lattice of dipoles, each of which takes only two positions and interacts only with its nearest neighbors. This model is attracting great interest in connection with the theory of phase transitions of the second kind. It is argued that phase transitions in binary alloys and with changes of crystal symmetry, and also the behavior of substances near the critical point, are described by this model.[1,2] Therefore it is interesting to ascertain how sensitive the results are to the form of the model, and in particular whether there are changes of the nature of the singularity in macroscopic quantities and of the shape of the correlation function when interactions with nonnearest neighbors are included.

In the present paper we consider a two-dimensional lattice, and include in addition to the interaction of nearest neighbors an interaction of certain nonnearest neighbors.

## 2. CALCULATION OF THE FREE ENERGY

Let us consider a two-dimensional lattice of the Ising type, consisting of two kinds of "atoms" which are arranged in a checkerboard pattern and interact with each other in the way shown in Fig. 1. The interaction energy between different atoms, i.e., along vertical and horizontal directions, is $-J_1$, and that along the diagonals is

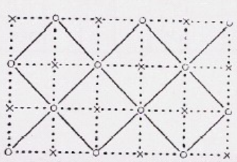


FIG. 1

$-J_2$. The difference between this model and the ordinary Ising lattice is that besides the interaction between nearest neighbors there is also an interaction along the diagonals for the atoms of one kind. For $J_1 = 0$ the system goes over into an ordinary Ising lattice.

The partition function is given by the expression

$$Z=\sum_{(\sigma)}\exp\Big[\frac{J_1}{T}\sum_{k,l=1}^{L}\sigma_{kl}(\sigma_{kl+1}+\sigma_{k+1l})+\frac{J_2}{T}\sum_{k,l=1}^{L}\eta_{kl}\sigma_{kl}(\sigma_{k+1,l+1}+\sigma_{k-1,l+1})\Big],$$

$$\sigma_{kl}=\pm1,\qquad \eta_{kl}={}^1/_2[1+(-1)^{k+l}], \qquad (1$$

where L is the number of atoms in a row or column. The expression (1) can be put in the form

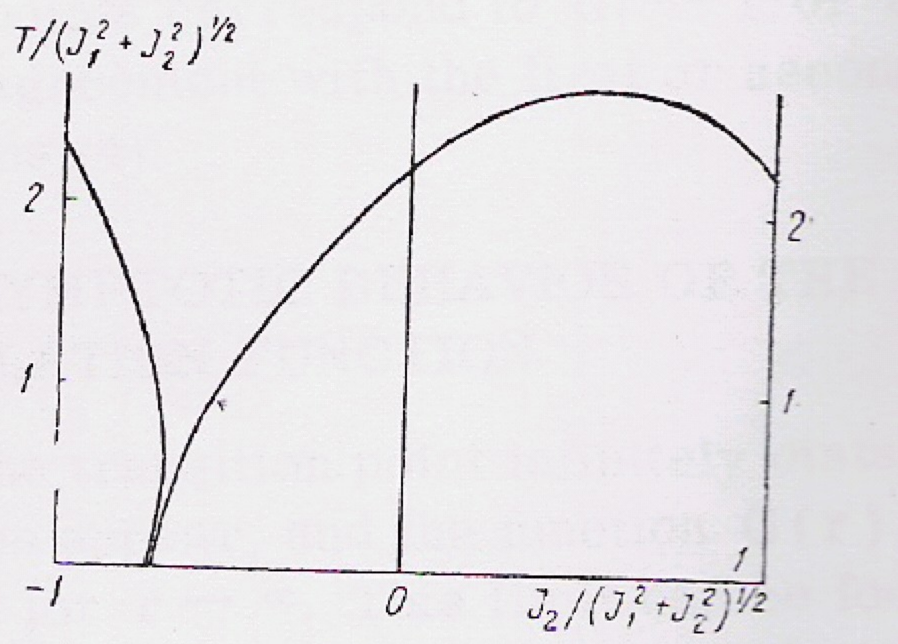
$$Z=(1-x^2)^{-N}(1-y^2)^{-N/2}S,$$

$$S=\sum_{(\sigma)}\prod_{k,l}(1+x\sigma_{kl}\sigma_{kl+1})(1+x\sigma_{kl}\sigma_{k+1l})(1+y\eta_{kl}\sigma_{kl}\sigma_{k+1,l+1})\times(1+y\eta_{kl}\sigma_{kl}\sigma_{k-1,l+1}). \qquad (2)$$

Here $x = \tanh(J_1/T)$, $y = \tanh(J_2/T)$, and $N = L^2$ is the total number of atoms. The quantity S is a polynomial in x and y, in which the coefficient $g_{nm}$ of $x^ny^m$ is equal to the number of ways closed polygons can be constructed in which the total number of vertical and horizontal links is n and the total number of diagonal links is m (cf., e.g.,[3]).

It is shown in a paper by Vdovichenko[4] that for the ordinary Ising lattice the quantity $g_{nm}$ can be put in the form of a sum over closed loops, each loop being taken with the factor $(-1)^r$, where r is the number of intersections. Our present case differs from the usual one by the fact that there can be intersection not of only two,



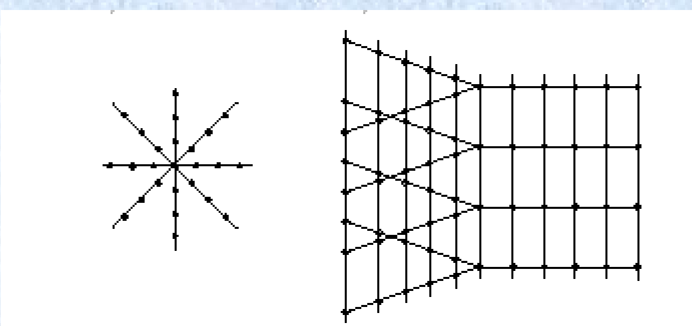


# Local phase transition: spherical model

$$H = - J \sum_{<i,j>} S_i S_j \qquad \left( \sum_i S_i^2 = N \right)$$

$$1 = \frac{k_B T}{2N} \sum_\alpha \frac{1}{\mu - \frac{J}{2} e_\alpha}$$

← eigenvalues of the adjacency matrix



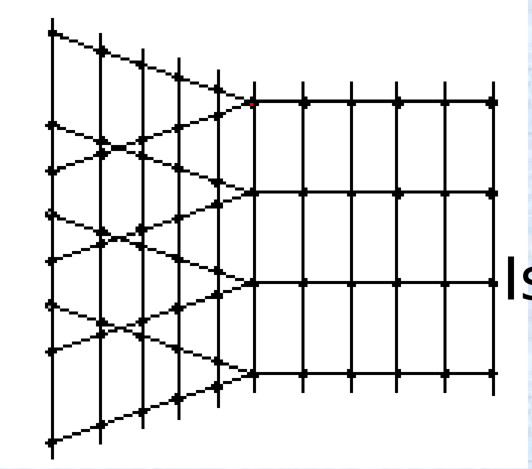
$$M_i = \langle S_i \rangle \propto \sqrt{1 - \frac{T}{T^*}} e^{-i/\eta}$$

$$k_B T^* = J \frac{p-2}{\sqrt{p-1}}$$

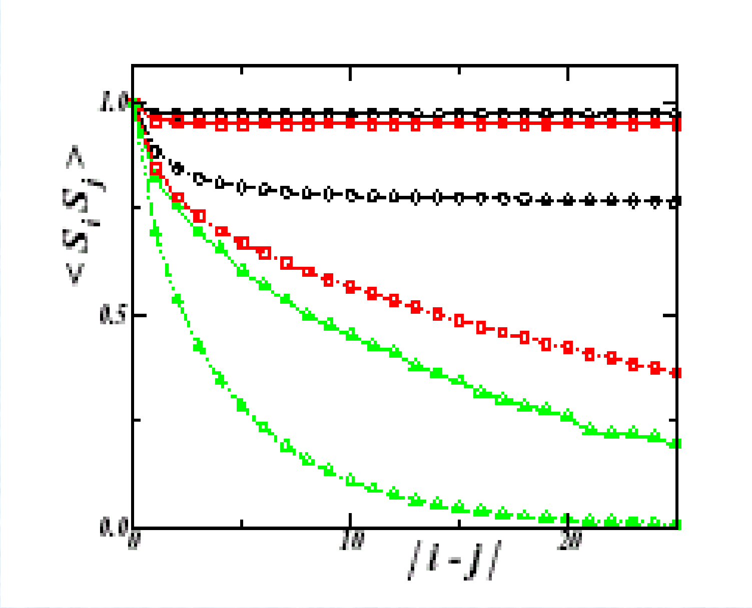
Of course $M = \frac{1}{N} \sum_i \langle S_i \rangle = 0$ for finite temperature $(T_c = 0)$

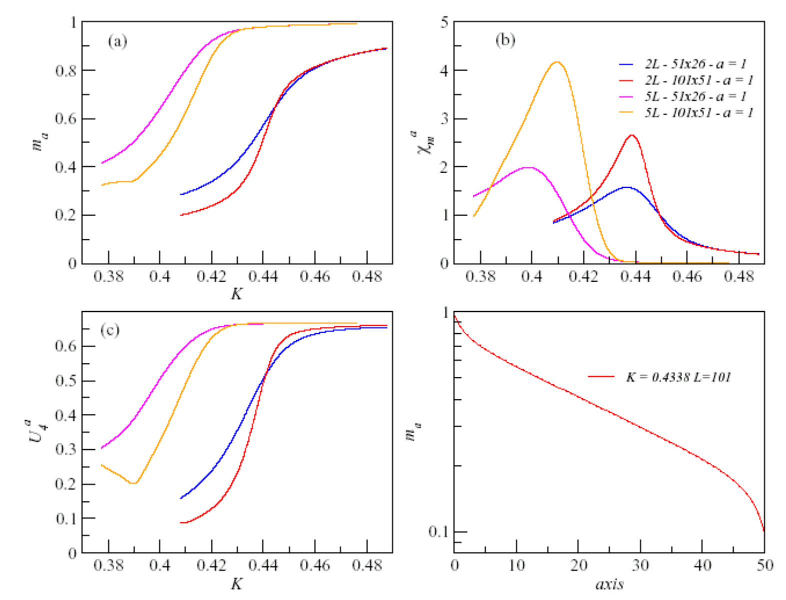
R. Marchetti, M. Rasetti, A. Trombettoni, and P. Sodano, submitte

# Classical statistical mechanics models with inhomogeneities: Ising model



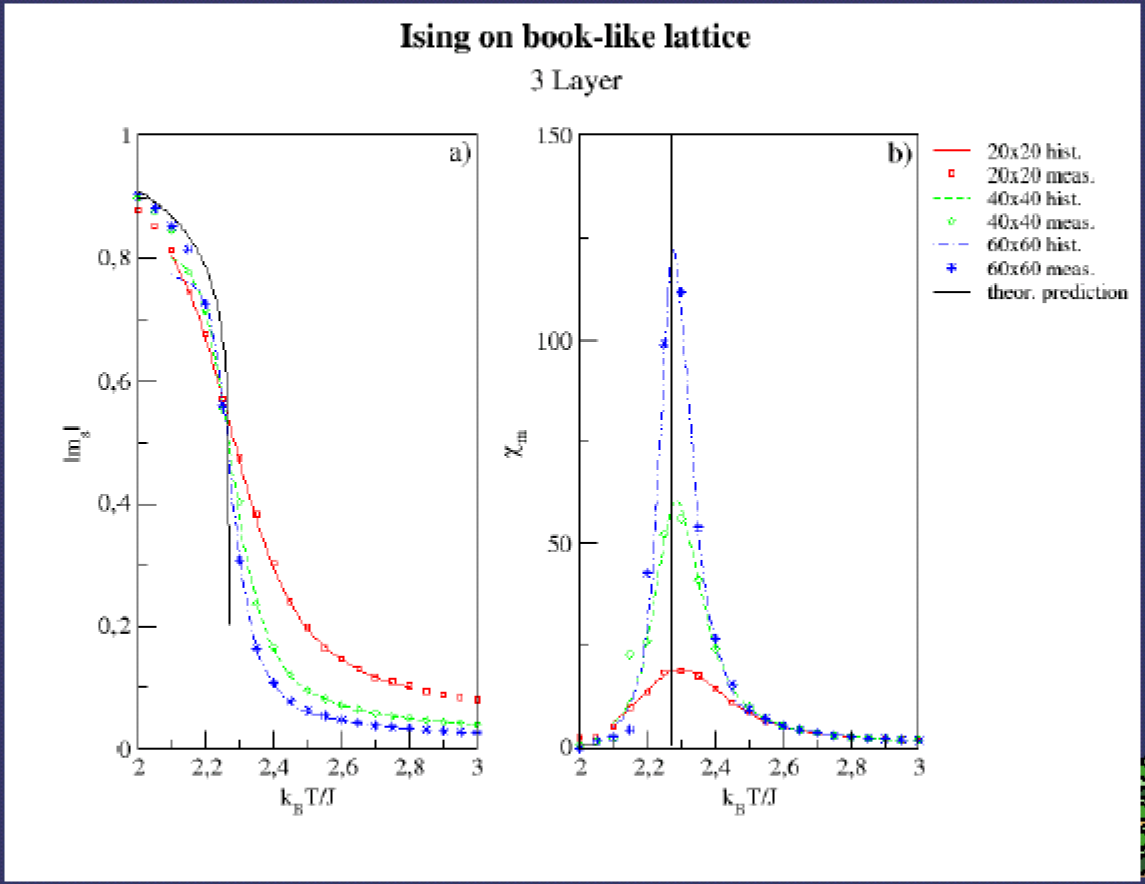
Ising model on a book graph



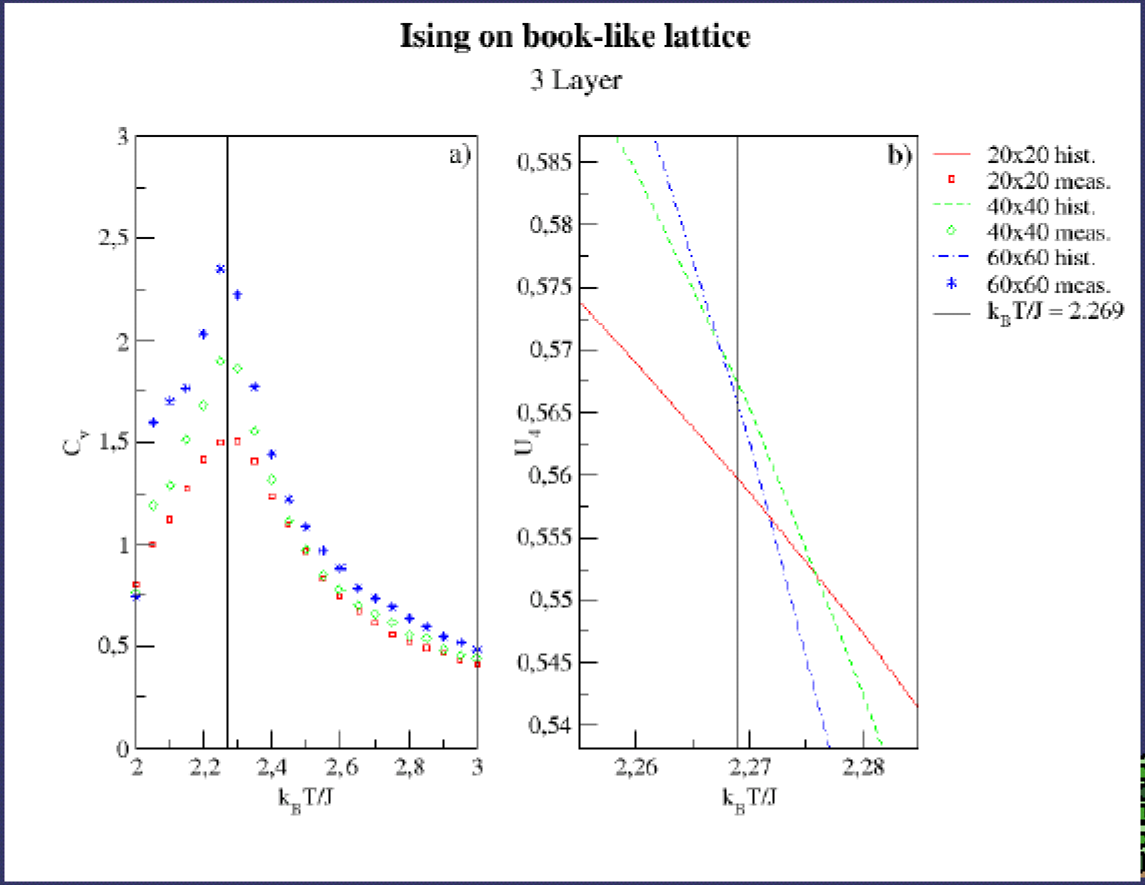
# Ising Model on Book Graph

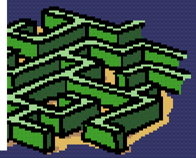
*Global Magnetization and Susceptibility*




# Ising Model on Book Graph

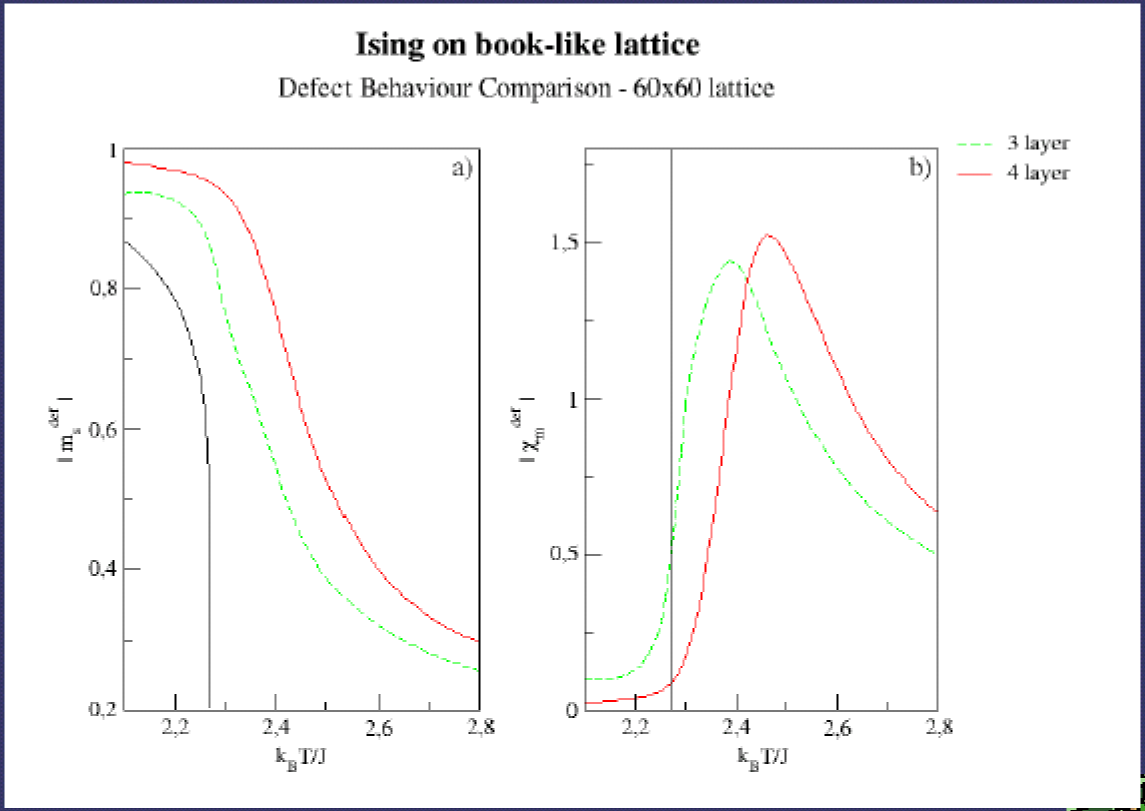
## *Specific Heat and Binder Cumulant*



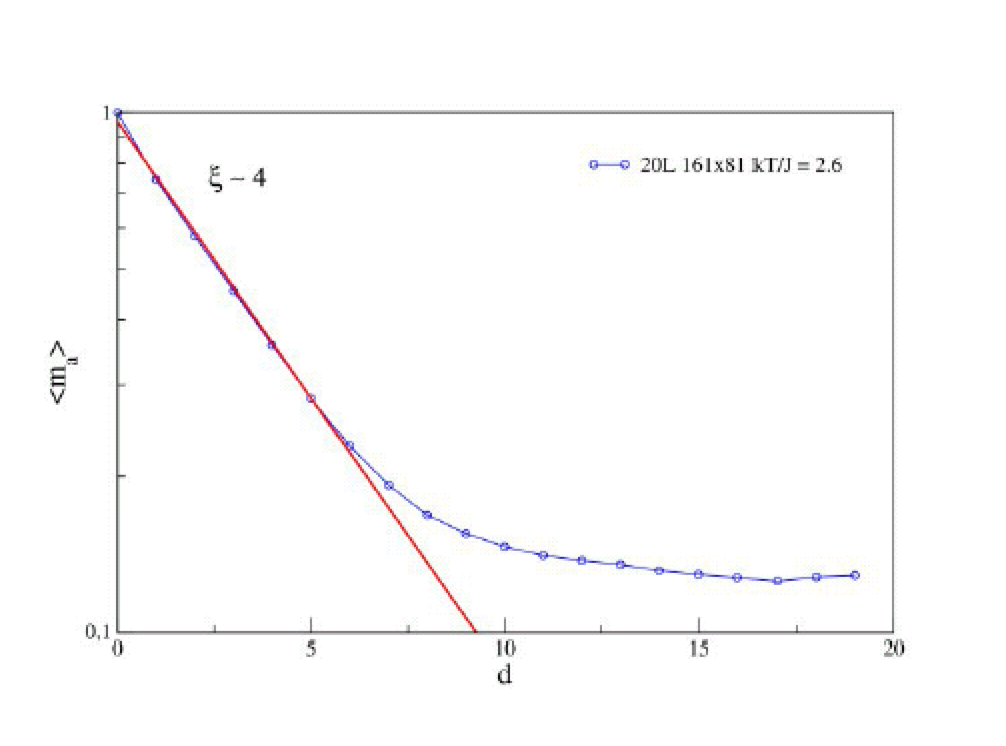
# Ising Model - Defect Behaviour

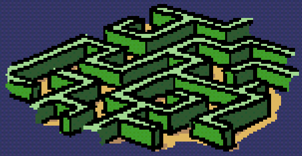
*Magnetization and Susceptibility*




# Ising Model - Single Axis Behavior

*Magn. as a function of the Distance from the Preferred Axis*

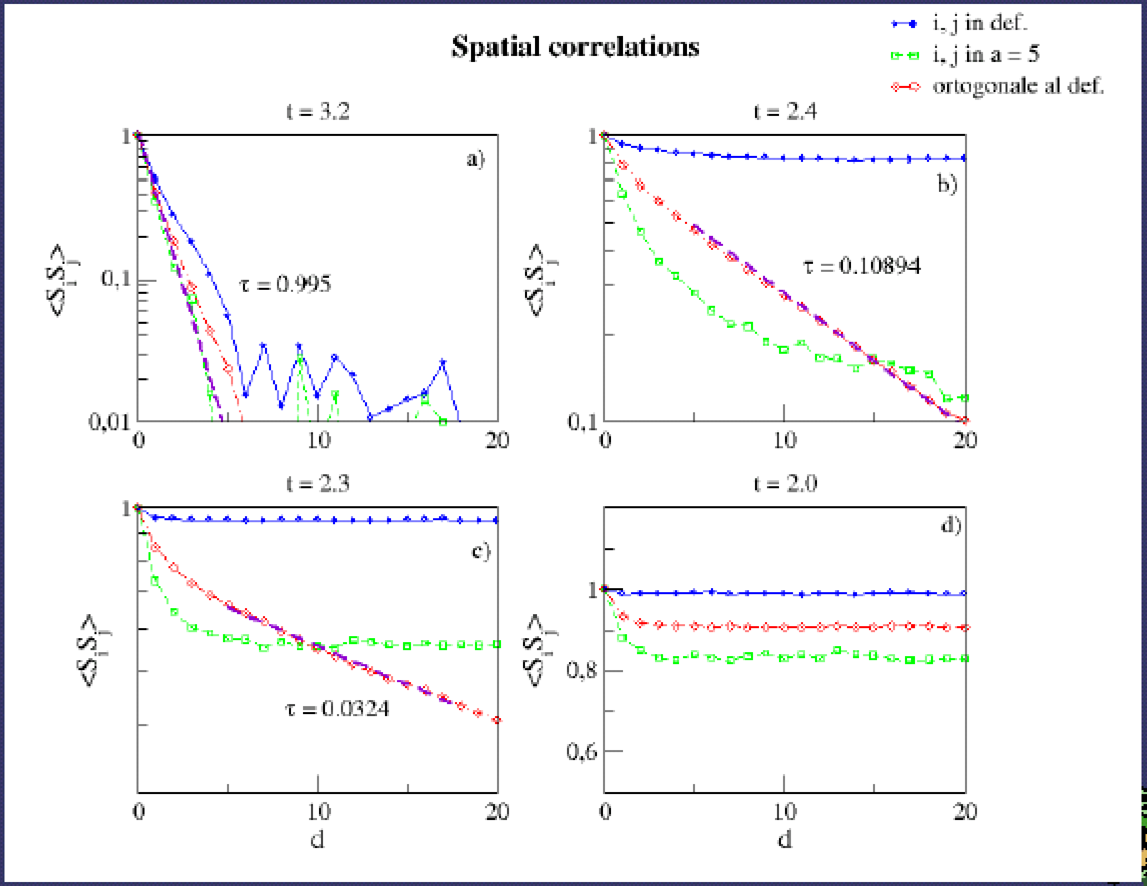


Exponential decay of axis magnetization per site as a function of distance from preferred axis above the global critical temperature $T_c$

# Ising Model

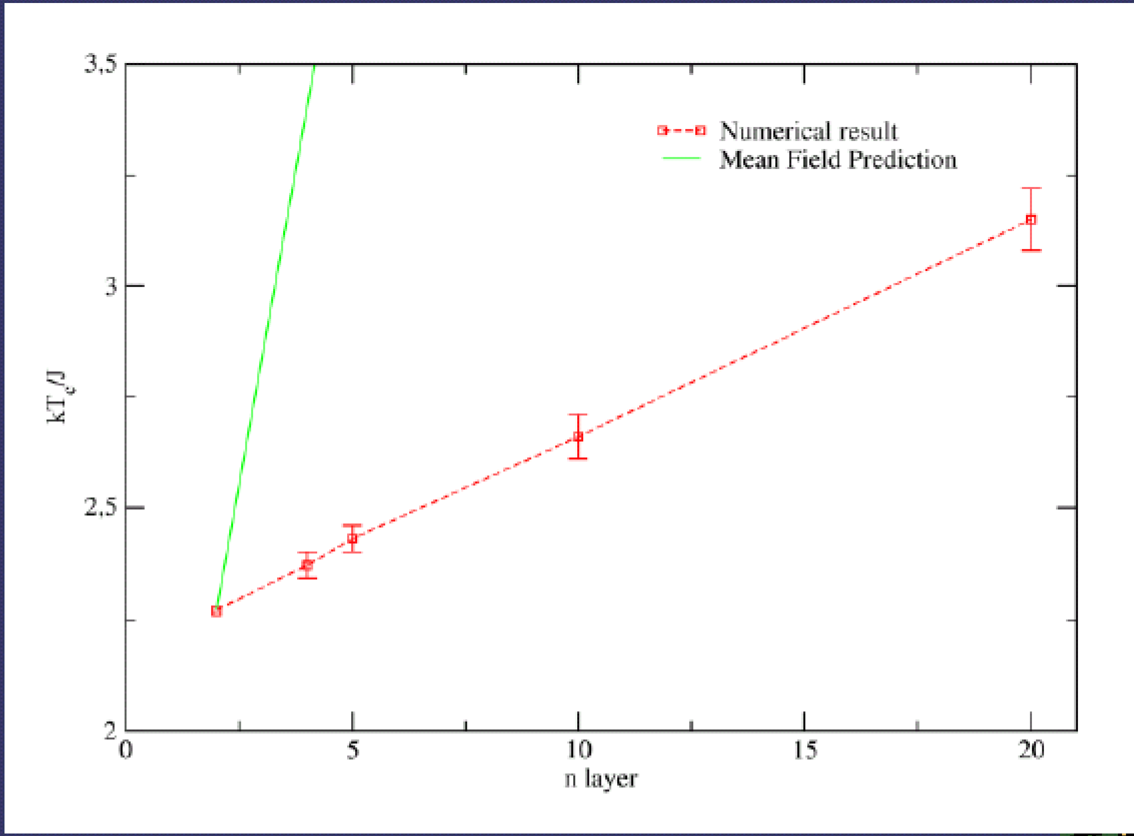
## *Spatial Correlation*




$t = k_B T / J$

# Ising Model - Single Axis Behavior

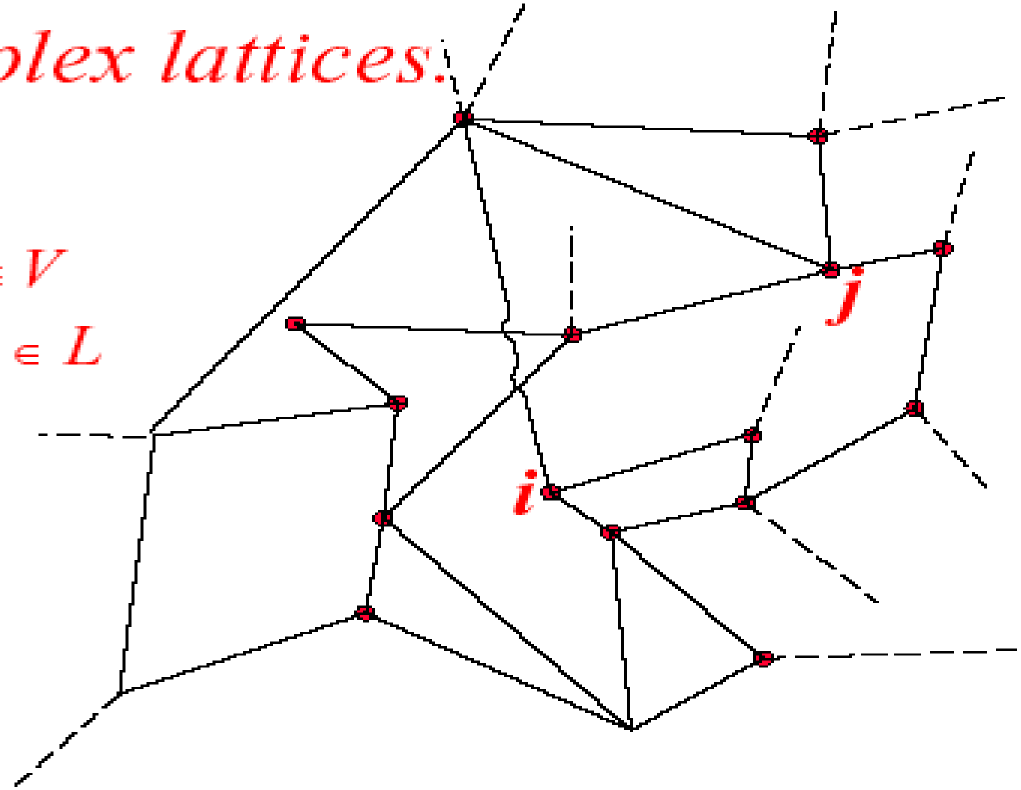
*"Defect Critical Temperature" - function of p*



*Some tools to describe complex lattices:*

Set of sites $i \in V$

Set of links $i\text{-}j \in L$




Adjacency matrix $A_{ij} = \begin{cases} 1 \text{ if } i - j \text{ is a link} \\ 0 \quad \text{otherwise} \end{cases}$

Coordination number $z_i = \sum_j A_{ij}$

Chemical distance $d_{ij}$

(shortest path from *i* to *j*)

# Energy spectrum

Formed by $N_s$ states and divided in 3 parts: { $E_0$ , $\sigma_0$ , $E_+$ }

$\sigma_0$ → $pL$-1 delocalized states with $E \in [-2t, 2t]$

**Density of states:**

$$\rho(E) = \sum_n \delta(E - E_n) \longleftrightarrow \rho(E) = \frac{1}{\pi\sqrt{4t^2 - E^2}}$$

two bound states → $E_0 < -2t$ *and* $E_+ > 2t$

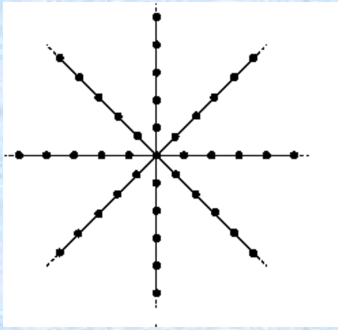
**($E_0$ : localized ground-state)**

$$E_0 = -t\frac{p}{\sqrt{p-1}}$$

$p=2$ → $E_0 = -2t$ ***Linear chain***

$p \neq 2$ → $\Delta(p) = |E_0| - 2t$ ***Gapped Spectrum***

# Thermodynamics for bosons hopping on a star lattice



$$N_T = N_{E_0} + \int_{E\in\sigma_0} \frac{N_S\,\rho(E)}{z^{-1}e^{\beta(E-E_0)}-1} + N_{E_+}$$

Ground-State — Delocalized — Excited

$$f = \frac{N_T}{N_S}$$

$$E_J = 2tf$$

$$T < T_C \quad \rightarrow \quad \frac{N_{E_0}}{N_S} \neq 0$$

$$T_C \approx \frac{p-2}{2\sqrt{p-1}}\,\frac{E_J}{k_B}$$

**Topology effect**

with the interwell barrier $V_0 \approx 2\pi\,\hbar$ $50\ KHz$ and filling $f \approx 200$ then: $E_J \approx 50\ nK$

$$\frac{N_{E_0}\left(T/T_C\right)}{N_T} \approx 1 - \frac{T}{T_C} \qquad L \rightarrow \infty$$

Typical of one-dimensional condensate [see W. Ketterle and N. J. van Druten, *Phys. Rev. A* **54**, 656 (1996)]

I. Brunelli *et al.*, *J. Phys. B* **37**, S275 (2004)