Finite-size resource scaling for learning quantum phase transitions with fidelity-based support vector machines

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Abstract: Via quantum kernels, the learning cost—defined by shot measurements S_spread—must exceed the kernel's minimal resolution, a.k.a. « spread ». This cost increases with Hamiltonian symmetries, making phase transitions more difficult to learn.

Fidelity-based quantum kernels

$$K(x_i, x_j) = F = \left|\langle\psi(x_i)\middle|\psi(x_j)\rangle\right|^2$$

provide a natural way to detect quantum phase transition in many-body physics from the similarity between ground states, without assuming an order parameter beforehand [2,3]. In practice, however, these kernels must be estimated on quantum hardware from a finite number of measurements, so the key question is not only whether the classifier works, but also how the required measurement cost scales with system size and with the symmetry of the underlying model. Our goal is precisely to clarify this resource-scaling problem for fidelity-based support vector machines [4] and to identify which physical features make a model easier or harder to learn in a quantum-kernel setting. To do so, we study a tunable one-dimensional spin-1/2 chain that interpolates among the Ising, XY, XX, and XXZ regimes,

$$H = -\sum_{i=1}^{N}\left[\frac{1+\gamma}{2}(\sigma_i^x\sigma_{i+1}^x) + \frac{1-\gamma}{2}(\sigma_i^y\sigma_{i+1}^y) + \Delta(\sigma_i^z\sigma_{i+1}^z)\right] - h\sum_{i=1}^{N}\sigma_i^z$$

where γ controls the XY anisotropy, Δt is the zz interaction, and h is the transverse field.

In our kernel analysis, we address two main issues. First, the kernel entries estimated from a finite number of SWAP-test shots are affected by measurement error. If this error exceeds the natural spread of the kernel entries, the informative structure of the kernel becomes blurred. To prevent this, we derive a bound that quantifies the minimum number of shots required to resolve the spread of the kernel values.

$$S_{\text{spread}} \geq \frac{1-k_{\text{repr}}^2}{(1-P_{\text{spread}})\epsilon^2\Delta_{\text{ensemble}}^2}, \Delta_{\text{ensemble}} = Q_3 - Q_1,$$

where Δ_{ensemble} denotes the interquartile spread of the kernel entries and k_{repr} is a representative kernel value P_{spread} is the chosen confidence level, while ϵ sets the maximum tolerated relative error in resolving the kernel spread[5,6].

Second, we consider the tendency of the kernel values to concentrate near zero. In fidelity-based kernels, this effect becomes more pronounced as the system size N increases, since the overlap between many-body ground states typically decreases, driving the kernel entries toward zero. To avoid this concentration effect, we derive a second bound.

$$S_{\text{CA}} \geq \frac{1-k_{\text{repr}}^2}{(1-P_{\text{CA}})\epsilon_{\text{CA}}^2 k_{\text{repr}}^2},$$

Likewise, P_{CA} is the chosen confidence level, while ϵ_{CA} sets the maximum tolerated relative error in avoiding concentration around the representative kernel value.

To evaluate these quantities across the different regimes, we compute the fidelities analytically for the free-fermion Ising/XY/XX family through the Bogoliubov-angle representation,

$$\theta_q(h) = \frac{1}{2}\arctan\left(\frac{\gamma\sin q}{h-\cos q}\right),$$

Which gives

$$F(h_a, h_b) = \prod_{q>0} \cos^2 (\theta_q(h_b) - \theta_q(h_a)),$$

whereas for the interacting XXZ chain we obtain them by exact diagonalization [7]. As a result, we find a clear hierarchy in the measurement cost: moving from the Z_2-symmetric Ising/XY regimes to the more symmetric XX/XXZ regimes enhances kernel concentration and therefore increases the number of shots required for reliable phase classification [7,8].

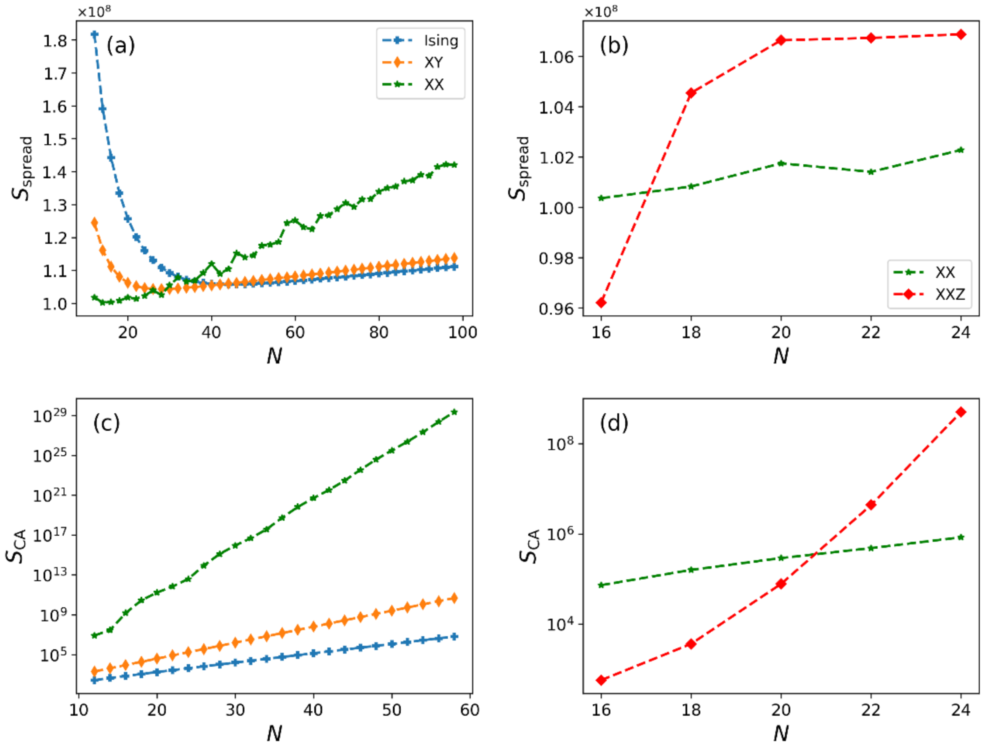

Fig. 1. Finite-size measurement costs for estimating fidelity-based quantum kernels with the SWAP test. The shot requirements are quantified by the spread bound S_{spread} and the concentration-avoidance bound S_{CA}. For Ising, XY, and XX, the training points are sampled from $h \in [0.7, 0.95] \cup [1.05, 1.3]$ at $\Delta = 0$, with $\gamma = 1$(Ising), $\gamma = 0.5$(XY), and $\gamma = 10^{-3}$(XX). For XXZ, the training points are sampled from $\Delta \in [0.35, 0.45] \cup [0.55, 0.65]$ at $h = 0$ and $\gamma = 10^{-3}$. Panels (a) and (c) show the Ising/XY/XX cases, while panels (b) and (d) compare XX and XXZ. In all panels we use $\epsilon = 10^{-3}$, $P_{\text{spread}} = 0.99$, $\epsilon_{\text{CA}} = 10^{-3}$, and $P_{\text{CA}} = 0.99$. The results show that higher symmetry leads to stronger kernel concentration and therefore larger shot requirements.

Example References

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