Number of steady and asymptotic states of quantum evolutions

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Abstract

We will discuss sharp upper bounds on the number of linearly independent steady and asymptotic states of discrete- and continuous-time Markovian quantum evolutions. Importantly, the bounds depend only on the dimension of the system.

Let us consider an open quantum system described by a finite-dimensional Hilbert space $\mathcal{H}$ with $d = \dim(\mathcal{H})$. Let $\mathcal{S}(\mathcal{H})$ be the set of density operators on $\mathcal{H}$, i.e. the states of the system under account.

The dynamics in a fixed time interval $[0, \tau]$, with $\tau > 0$, is described by a quantum channel Φ, viz. a completely positive trace-preserving map on the space $\mathcal{B}(\mathcal{H})$ of linear operators on $\mathcal{H}$. If the state of the system at time $t = 0$ is $\rho(0) \equiv \rho$, its discrete-time evolution at time $t = n\tau$, with $n \in \mathbb{N}$, will be given by the action of the n-th fold composition Φ^n of the map Φ, i.e. $\rho(n\tau) = \Phi^n(\rho)$, with $n \in \mathbb{N}$.

The asymptotic evolution, which corresponds to the limit $n \to \infty$, takes place within the attractor subspace [1]

$$\mathrm{Attr}(\Phi) = \mathrm{span}\{X \in \mathcal{B}(\mathcal{H}) \,|\, \Phi(X) = \lambda X \text{ for some } |\lambda| = 1\}. \tag{1}$$

In particular, any evolved state $\Phi^n(\rho)$ asymptotically moves towards one of the states $\rho_{\mathrm{as}} \in \mathrm{Attr}(\Phi)$, which we will call the asymptotic states of Φ. Moreover, the attractor subspace contains the fixed-point subspace

$$\mathrm{Fix}(\Phi) = \{X \in \mathcal{B}(\mathcal{H}) \,|\, \Phi(X) = X\}, \tag{2}$$

involving the steady states of Φ, i.e. $\Phi(\rho) = \rho \in \mathcal{S}(\mathcal{H})$.

In this talk we will focus our attention on the dimensions $\ell_0 = \dim(\mathrm{Fix}(\Phi))$ and $\ell_{\mathrm{P}} = \dim(\mathrm{Attr}(\Phi))$ of the fixed-point and attractor subspaces of the evolution induced by Φ, respectively. Importantly, ℓ_0 (ℓ_{P}) represents the number of linearly independent steady (asymptotic) states of the dynamics [2]. If Φ is a unitary channel different from the identity 1, namely $\Phi(\cdot) = U \cdot U^\dagger$ with U non-scalar unitary on $\mathcal{H}$, then the system is closed, and we find that [2]

$$\ell_0 \leqslant d^2 - 2d + 2, \quad \ell_{\mathrm{P}} = d^2, \tag{3}$$

while if Φ is not unitary, viz. the system is purely open, we obtain [2]

$$\ell_0 \leqslant \ell_{\mathrm{P}} \leqslant d^2 - 2d + 2. \tag{4}$$

Interestingly, the bounds (3) and (4) are sharp, and they only depend on the dimension d of the system. Similar bounds to (3) and (4) may be obtained for continuous-time Markovian evolutions [2].

References

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