C*-independence for Z_2-graded C*-algebras

Maria Elena Griseta[1]

1. University of Bari, Department of Mathematics, via Orabona 4, 70126 Bari, Italy

Abstract: We highlight some novel results for statistical independence for superalgebras of bounded operators on a Hilbert space. Namely, we characterize the so-called C*-independence and provide other notions of statistical independence for graded von Neumann algebras.

The notion of C*-independence or statistical independence was first introduced in [1], for its relevance in quantum field theory. Two C*-subalgebras A_1 and A_2 of a given C*-algebra A are said statistical independent when any two marginal states on A_1 and A_2 respectively, admit a common extension on A.

In addition, the Schlieder condition, sometimes called (S)-independence, *i.e.* $xy \neq 0$ for given non-vanishing elements x in A_1 and y in A_2 of C*-algebras A_1 and A_2, is a necessary condition for statistical independence [2]. In [3] it was shown by Roos that the (S)-independence is also a sufficient condition for statistical independence if the subalgebras A_1 and A_2 commute elementwise.

C*-independence has a counterpart in von Neumann algebras by the so-called W*-independence, which requires the natural condition of normality for the marginal states and their common extension. It is known that W*-independence is stronger than C*-independence.

Recall that a $\mathbf{Z}_2$-grading is obtained by assigning an involutive $*$-automorphism on a C*-algebra, and algebras without grading are seen as particular graded algebras with trivial (*i.e.* the identity map) automorphism. The grading uniquely splits the algebra into the direct sum of an even (*i.e.* grading-invariant) and odd parts.

Here [3], we analyze the C*-independence for superalgebras (*i.e.* $\mathbf{Z}_2$-graded C*- algebras) and its relationship with W*-independence for $\mathbf{Z}_2$-graded von Neumann algebras. In the $\mathbf{Z}_2$-graded setting, the notion of C*-independence and W*-independence are respectively expressed by the existence of a simultaneous even extension of two given even (*i.e.* grading-invariant) states on C*-subalgebras or even and normal states for von Neumann subalgebras.

We first introduce a suitable version of (S)-independence, tailored on the grade structure which generalizes the usual one in the case of trivial grading. This condition states that the twisted product of a couple of non-vanishing elements of the superalgebras A_1 and A_2 is ever non-zero.

Our main theorem states that this generalization of (S)-independence is equivalent to C*-independence once the two given $\mathbf{Z}_2$-graded C*-algebras A_1 and A_2 *commute with grading*. This means that x in A_1 and y in A_2 anti-commute if they are both odd, and commute otherwise. As this condition reduces to commutation for the trivial grading, we thus achieve a generalization of Roos' theorem.

As for W*-independence for $\mathbf{Z}_2$-graded von Neumann algebras, we finally show that it is stronger than C*-independence. As in the case of trivial grading, this is obtained by passing through intermediate other notions of statistical independence for graded structures.

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References

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